This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

**Usage guidelines**

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

**About Google Book Search**

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

# ELEMENTARY

# ANALYTICAL GEOMETRY.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

# ELEMENTARY
# ANALYTICAL GEOMETRY

## *FOR SCHOOLS AND BEGINNERS*

BY THE REV.

THOMAS G. VYVYAN, M.A.

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE, AND
MATHEMATICAL MASTER OF THE CHARTERHOUSE.

BIBLIOTHECA BODLEIANA

CAMBRIDGE:
DEIGHTON, BELL, AND CO.
LONDON: BELL AND DALDY.
1867.

183. f. 6.

THIS work is intended for beginners: that is for those who have acquired some little familiarity with algebraical working, and know some elementary trigonometry, and the forms of the conic sections.

The method of interpreting geometry by analysis has been used throughout instead of the method, perhaps more logical, but certainly more difficult, of taking the general equations, and interpreting them geometrically.

The discussion of the general equation of the second degree has therefore been omitted, but the equation is obtained in the last chapter from the definition of a conic section, and the case discussed in which the axes are tangents.

Symmetry of equations and of notation has been studied throughout the book.

In the ellipse, considerable prominence has been given to the excentric angle.

The examples have not been selected from Cambridge papers, as these are generally too hard for beginners: those at the beginning of each set are mere applications of the book-work, and many have, by Mr Walton's kind permission, been selected from his "*problems in plane coordinate geometry.*"

Trilinear coordinates have not been employed, since they now form a subject by themselves, and require a greater knowledge of algebra than is generally possessed by beginners.

A collection of problems, selected chiefly from Senate-House and College papers, has been placed at the end of the book.

A list of formulæ has also been appended: the reader is advised, *after* working through each chapter, to commit to memory those formulæ in the list which have been worked out in the course of that chapter.

CHARTERHOUSE, *Jan.* 31, 1867.

# TABLE OF CONTENTS.

---

## CHAPTER I.

### THE POINT.

ART. PAGE

1—5. Explanation of terms ........ 1

6—8. Equation to a curve: locus of an equation ........ 5

9. Distance of a point from the origin ........ 8

10. Distance between two points ........ 9

11. Coordinates of the point of bisection of a given line ........ 11

12. Area of a triangle whose angular points are given ........ 12

Example I. ........ 13

## CHAPTER II.

### TRANSFORMATION OF COORDINATES.

13. Origin, not axes, changed ........ 16

14. Axes not origin, axes rectangular ........ 17

16. Rectangular to polar ........ 18

17. One oblique to another ........ 20

18. Oblique to polar ........ 21

Example II. ........ 24

## CHAPTER III.

### The Straight Line.

ART. PAGE

20. Equation to a straight line in terms of the intercepts on the axes 26

21, 22. $Ax+By+C=0$ represents a straight line........................ 28

23. Equation to straight line making a given angle with the axis of $x$........................................................... 30

24. Through two given points........................................... 32

25. Parallel to a given line............................................... 32

26. In terms of perpendicular from origin and angles made by it with the axes................................................... 33

27, 28. Through a given point making given angle with a given line.................................................................. 34

29, 30. Polar Equations........................................... 37

31, 32. Comparison of Equations. Examples........................... 39

Example III.................................................. 45

## CHAPTER IV.

### The Straight Line (Continued).

33. Distance of a given point from a given line...................... 49

34. Between two parallel straight lines................................ 52

35. Angle between two straight lines.................................. 53

36. Point of intersection of given straight lines...................... 53

37. Condition that three points should be on the same straight line 54

38. Equation to a straight line through the intersection of two given straight lines..................................................... 55

39. Condition that three straight lines should pass through a point 56

Example IV.................................................. 58

## CHAPTER V.

### The Circle.

ART. PAGE

41—43. Equations: axes rectangular ........ 60
44. Conditions that an equation may represent a circle ........ 65
45, 46. Equation: axes oblique ........ 66
47, 48. Polar coordinates ........ 67
Example V. ........ 68
49—51. Tangent and normal—Definitions ........ 71
52—54. Equations ........ 71
55, 56. Conditions that a straight line should be a tangent ........ 73
57. Length of tangent from given point ........ 75
Example VI. ........ 76
58—64. Poles and Polars ........ 78
65—68. Two or more circles; radical axes and centre ........ 83
69, 70. Common tangents ........ 86
Example VII. ........ 89

## CHAPTER VI.

### The Parabola.

71. Definitions of Conic Sections ........ 94
72. Polar equation to a conic section ........ 94
73. Equation to the parabola; axes rectangular ........ 96
74—76. Form of curve: focal distance: latus rectum ........ 97
77, 78. Equation to tangent at $(x_1y_1)$ ........ 98
79. Tangent making angle $\theta$ with the axis ........ 99
80. Point of contact of the tangent $y=x\tan\theta+a\cot\theta$ ........ 100
81—83. Normals ........ 101

ART. PAGE
84—88. Properties of tangent: poles and polars ........................ 103
89, 90. Diameters ........................................................ 105
91. Curve referred to diameter and its tangent ........................ 107
92, 93. Polar equation to curve and tangent... ........................... 108
Example VIII.................................................... 109

## CHAPTER VII.

### The Ellipse.

94. Definition and equation.................................................. 116
95—97. Form of curve: centre: diameters.............................. 117
98. Focal distances of a point............................................. .. 119
99. Auxiliary circle ................................................. ..... .. 120
100. Excentric angle ........................................................ 121
101, 102. Polar equation; latus rectum .................................. 122
Example IX. ........................................................ 123
103. Equation to tangent at point $(x_1 y_1)$.................................. 126
104. in terms of excentric angle ................. 126
105—108. Perpendiculars from the centre and foci ................... 127
109. Condition that given line should be a tangent ...................... 129
110—112. Equations to normal ............................................. 130
113. Normal bisects the angle between the focal distances ............ 132
Example X. ........................................................ 133
115—120. Diameters; conjugate diameters ............................ 135
121, 122. Circumscribing Parallelogram.................................. 139
123. Equation referred to conjugate diameters ........................ 141
124. to tangent ................ 142
125, 126. Supplemental Chords.............................................. 143
Example XI. ........................................................ 144

## CHAPTER VIII.

### THE HYPERBOLA.

ART. PAGE
127—129. Definition and equation .......... 147
130, 131. Form of the curve .......... 148
132, 133. Definition of and equation to asymptotes .......... 150
134. Polar equation; centre pole .......... 152
135. focus pole; tracing the curve .......... 152
136. Conjugate hyperbola .......... 155
137. Excentric angle .......... 156
138—140. Focal properties .......... 158
141—146. Properties of the tangent and normal .......... 159
147, 148. Conjugate diameters .......... 161
149—152. Hyperbola and conjugate referred to asymptotes .......... 162
153—156. Tangents; asymptotes axes .......... 164
157. Rectangular hyperbola .......... 167
Example XII. .......... 167
Example XIII. Rectangular hyperbola .......... 171

## CHAPTER IX.

### GENERAL PROPERTIES OF CONIC SECTIONS.

158. Polar equation to tangent .......... 173
159. General equation $Ax^2+2Bxy+Cy^2+2Dx+Ey+F=0$ .......... 174
160. The curve is an ellipse, parabola, or hyperbola, as $AC-B^2 > = \text{ or } < 0$ .......... 176
161. Equation referred to two tangents as axes .......... 176
162. Parabola referred to two tangents .......... 178

ART. PAGE
163. Any straight line through a pole is harmonically divided by the curve and the polar ........ 178
Example XIV. ........ 180

Miscellaneous Problems and Examples ........ 183
List of Formulæ ........ 199
Answers to the Examples ........ 203

# ERRATA.

| PAGE | LINE | |
|---|---|---|
| 9 | 4 | from bottom, for $ON - ON$ read $ON - OM$ |
| 12 | fig. | for Z read L |
| 18 | 4 | from bottom, for *rectagular* read *rectangular* |
| 22 | 2 | from bottom, for $+a^2$ read $=a^2$ |
| 65 | 9 | from top, for *to* read *and* |
| 82 | Art. 61 | for *If the polar...then Q* read *The polar...if Q* |
| " | " | for *Q lies on the pole of P* read *Q lies on the polar of P* |
| 91 | q. 15, | for *have three* read *may have four* |
| 116 | 18, | omit *Let* $OS=c$ |
| 130 | 12, | for $\left.\frac{\cos^2\theta}{b}\right)$ read $\left.\frac{\cos^2\theta}{b^2}\right)$ |
| 163 | 12, | for $x'y \sin \omega$ read $x'y' \sin \omega$ |
| 175 | 14, | for $p$ read $e^2p$ |
| 177 | 6, | for $Ax^2+2Dx+E=0$ read $Ax^2+2Dx+F=0$ |
| 180 | q. 6, | for $\left(\frac{x-a}{a}\right)$ read $\left(\frac{x-a}{a}\right)^2$. |

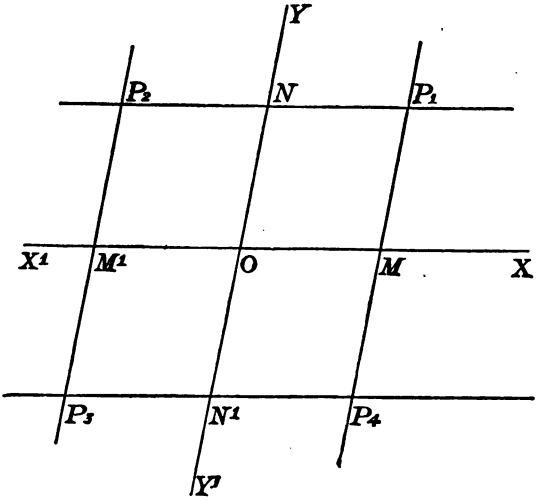
# ANALYTICAL GEOMETRY.

## CHAPTER I.

### THE POINT.

1. THE object of Analytical Geometry is to determine the algebraical equations by which certain well-known lines and curves may be represented, and from those equations to deduce their geometrical properties; and conversely, having given the equations, to determine the lines and curves which they represent.

2. First, it is necessary to shew how the relative distances of points in a plane may be represented by algebraical symbols.



Let $O$ be any point in the plane, through $O$ draw a straight line $X'OX$ of unlimited length, also through $O$ draw another straight line $YOY'$ making a fixed angle with $X'OX$; in $OX$ take any point $M$, and through $M$ draw a straight line parallel to $YOY'$: then it is evident that all points in this straight line are at the same distance from $YOY'$, and that if this distance be measured parallel to $XOX'$, it is equal to $OM$.

Let us denote that distance by $x$; then all the points on the straight line $P_1MP_4$ have the same value of $x$, and if we call that value $a$, the equation $x=a$ is true for all points on that line.

Similarly, if through any point $N$ on $YOY'$ we draw $P_1NP_2$ parallel to $XOX'$, the distance measured parallel to $YOY'$ of every point on this line from $XOX'$ is equal to $ON$; if then we call that distance $y$, all points on that straight line have the same value of $y$, and if $ON$ be denoted by $b$, the equation $y=b$ is true for all points on the straight line $P_1NP_2$.

Let these straight lines intersect in $P_1$; then at the point $P_1$, $x=a$, $y=b$.

**3.** If however we take points $M'$, $N'$, on $OX'$, $OY'$ such that $OM'=OM$, $ON'=ON$, and through $M'$, $N'$ draw straight lines parallel to $XOX'$, $YOY'$, it would seem that for each of the four points $P_1$, $P_2$, $P_3$, $P_4$, where these straight lines intersect, $x$ would be equal to $a$, and $y$ to $b$; it is necessary therefore to adopt some convention to distinguish between these four points.

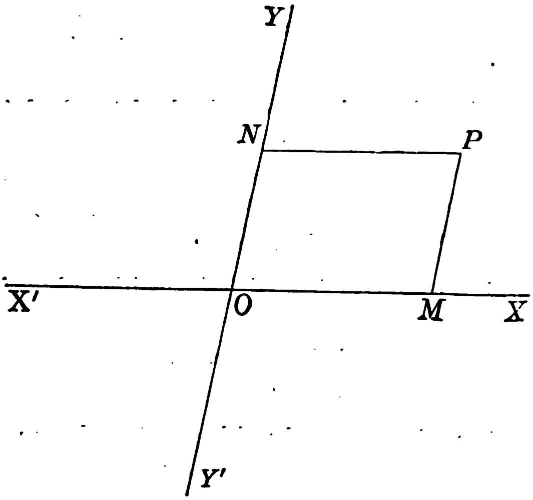
Let us suppose that all straight lines drawn in a definite direction be considered positive, those drawn in the opposite direction will evidently be negative.

Let all lines drawn parallel to $XOX'$ on the right of $YOY'$ be positive, then those drawn to the left will be negative; thus, if $OM=a$, $OM'=-a$.

Similarly, let all lines drawn above $XOX'$ parallel to $YOY'$ be considered positive, then those drawn below will be negative, and if, as before, $x, y$ be the distances of $P_1, P_2, P_3, P_4$ from $YOY'$, $XOX'$ respectively, measured parallel to $XOX'$, $YOY'$, we shall have, at $P_1$, $x=a$, $y=b$, at $P_2$, $x=-a$, $y=b$, at $P_3$, $x=-a$, $y=-b$, and at $P_4$, $x=a$, $y=-b$.

## 4. *Axes, Coordinates, System of coördinates.*

We can now explain what is meant when we speak of the coordinates of a point.



Let $O$ be a fixed point in a plane $XOY$; $XOX'$, $YOY'$ two fixed straight lines in the plane, $P$ any point in the plane. Through $P$ draw $PM$, $PN$ parallel to $OY$, $OX$ respectively cutting $OX$, $OY$ in $M$, $N$, then it is evident that if $PM$, $PN$

are known in *magnitude* and *direction* or *sign*, the position of the point $P$ is completely determined, and that, since $OM = PN$, if $OM$, $PM$ are known, $P$ is also known.

$OM$, $PM$ are called the *coordinates* of $P$, and are denoted by $x$, $y$, respectively; $O$ is called the *origin*, $OX$ the *axis* of $x$, since $x$ is measured along it, $OY$ the *axis* of $y$, since $y$ is measured parallel to it.

$OM$, $PN$ are said to be the coordinates of $P$ belonging to the system $YOX$.

The point $P$ whose coordinates are $x$, $y$, is often called the point $(xy)$: thus, a point, for which $x = a$, $y = b$, is called the point $(ab)$.

When the point $P$ is not completely determined, its coordinates are denoted by the variables $x$, $y$, $x'$, $y'$, $\xi$, $\eta$ &c. When the position of a point is completely known, the coordinates are generally denoted by the letters $a$, $b$, $h$, $k$, or by $x$, $y$ with suffixes such as $x_1$, $y_1$; $x_2$, $y_2$, &c.

Thus, if we want to determine the position of a point with reference to fixed points, we shall use $x$, $y$ for the coordinates of the *unknown* point; $a$, $b$; $h$, $k$; $x_1$, $y_1$, &c., for the *known* coordinates of the fixed points.

A system of axes may be either rectangular or oblique; that is, the angle $YOX$ may be either a right angle, or an oblique angle.

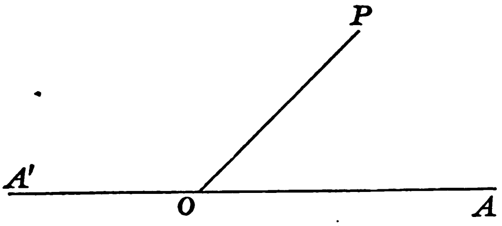
We shall, in future, always suppose the axes to be rectangular, unless the contrary is stated.

When the angle $YOX$ is not a right angle, it is generally denoted by $\omega$.

### 5. *Polar Coordinates.*

There is another kind of system of coordinates which is often useful. Let $O$ be, as before, a fixed point, $AOA'$ a fixed straight line, $P$ any point. Join $OP$.

It is evident that $P$ is known in position if we know its distance from $O$, and the angle that distance makes with $OA$.



Thus, if we denote the distance $OP$ by $r$, and the angle $POA$ by $\theta$, the position of $P$ is determined if $r$ and $\theta$ are known.

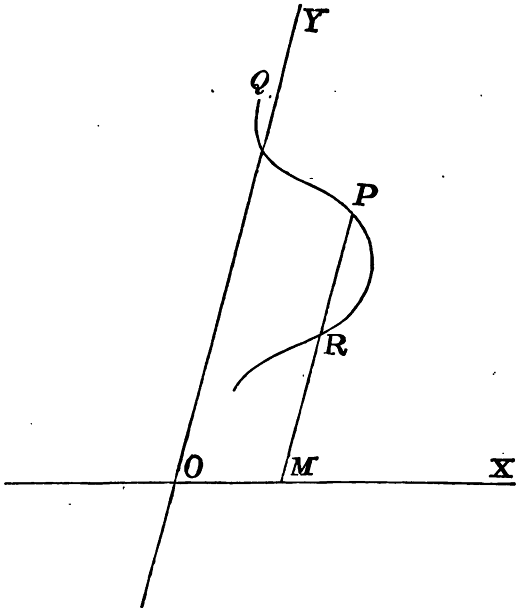
$r$, $\theta$ are the polar coordinates of $P$; $O$ is called the *pole*, $OA$ the *initial line*, $OP$ the *radius vector of P*.

As in Trigonometry, the $\angle\ POA$ is considered positive when measured in the direction opposite to that of the order of figures on a watch, negative when in that direction.

### 6. *Equation to a curve; Locus of an Equation.*

We will now explain how curves may be represented by indeterminate equations between the variables $x$, $y$, or $r$, $\theta$.

Let $QPR$ be any curve, then as we pass along it from point to point, it is evident that we get at each point different values of $x$ and $y$: that is, the values of $x$ and $y$ are not determined by saying that they belong to some point on the curve $QPR$.



If however we take a particular point $P$ on the curve such that $x$ is of determinate value, the value of $y$ becomes determinate also. There may, it is true, be more than one such value of $y$, but however many there are, they are determined by the fact of determining $x$.

Hence for every curve there must be some determinate relation between $x$ and $y$, which may be represented by an equation; this equation is called the equation to that curve.

The curve is said to be the *locus* of the points whose coordinates are connected by the equation, or, more shortly, the *locus of the equation.*

We shall hereafter determine the equations which belong to certain well-known lines and curves, and discuss the geometrical meaning of the indeterminate equations of the first and second degrees,

$$Ax + By + C = 0,$$

and

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

**7.** First, let us consider the meaning of equations in which only one of the variables $x$ and $y$ is involved.

Suppose we have the equation $x = a$; since the distance of all points represented by this equation from the axis of $y$ is constant, these points must lie on a straight line at that distance parallel to that axis: hence, $x = a$ is the equation to a straight line parallel to the axis of $y$.

Similarly, $y = b$ is the equation to a straight line parallel to the axis of $x$.

Next, suppose we have the quadratic equation,

$$x^2 + px + q = 0.$$

Let $a_1$, $a_2$, be the roots of this equation.

Then for every point on the locus of this equation, either $x = a_1$, or $x = a_2$, and therefore this point lies on one of the lines represented by the equations $x = a_1$, $x = a_2$. Similarly, since every equation involving one variable has as many roots as it has dimensions, it must represent a series of straight lines parallel to one of the axes, at distances from that axis equal to the different roots.

**8.** The position of a point may be defined not only *explicitly* by coordinates, but *implicitly*, by means of simultaneous equations, which these coordinates satisfy.

For we know from Algebra that if we have two simultaneous equations between two variables, we can find the roots of those equations, that is, the variables are not really indeterminate but have definite values, and therefore these values are the coordinates of known points.

Thus, suppose we have the equations

$$3x + 4y = 10,$$

$$x - y = 1\,;$$

then $$x = 2,\ y = 1,$$

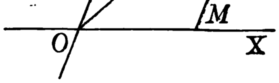
or $$x^2 + y^2 = 25,$$

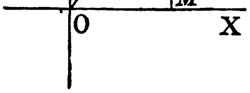
~~$x \overset{-}{+} y = 1$;~~

then $$x = 4,\ y = 3,\ \text{or}\ x = -3,\ y = -4.$$

We shall see hereafter, that these equations, taken *together*, represent the points of intersection of those curves, which the same equations taken *separately* represent.

9. *To find the distance of any point from the origin in terms of the coordinates of that point.*





First, let the coordinates be rectangular.

Then $$OP^2 = OM^2 + PM^2 = x^2 + y^2,$$
$$OP = (x^2 + y^2)^{\frac{1}{2}}.$$

Secondly, let the coordinates be oblique.

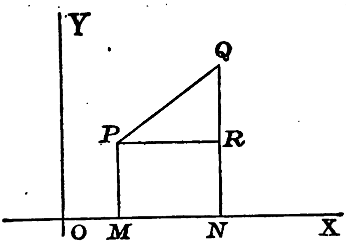
Then $$OP^2 = OM^2 + PM^2 - 2OM\,.\,MP \cos OMP$$
$$= OM^2 + PM^2 + 2OM\,.\,MP \cos YOM,$$
or $$OP = (x^2 + y^2 + 2xy \cos \omega)^{\frac{1}{2}}.$$

**10.** *To find the distance between two points.*

Let $\delta$ be the distance required.

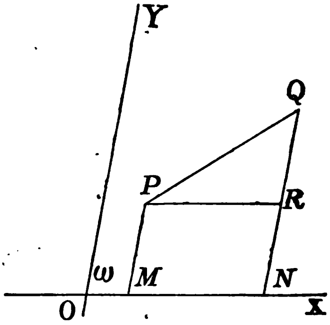
Let $P$, $Q$ be the points, and let their coordinates be $x_1\, y_1$, $x_2\, y_2$ respectively.

Draw $PR$ parallel to the axis of $x$ cutting $QN$ in $R$. And, first, let the axes be rectangular.



Then $$PQ^2 = PR^2 + QR^2.$$
But $$PR = MN = ON - ON = x_2 - x_1,$$
$$QR = y_2 - y_1,$$
$$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$
or $$\delta = \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}^{\frac{1}{2}}.$$

Next, let the axes be inclined at an angle $\omega$.



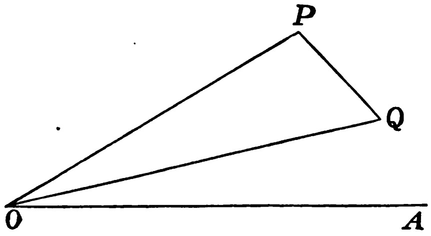
Then $PQ^2 = PR^2 + QR^2 - 2PR \,.\, QR \cos PRQ$,

but $PRQ = ONQ = \pi - \omega$;

$\therefore\ PQ^2 = PR^2 + QR^2 + 2PR \,.\, QR \cos \omega$,

or $\delta = \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos \omega\}^{\frac{1}{2}}$.

Next, let the coordinates be polar.

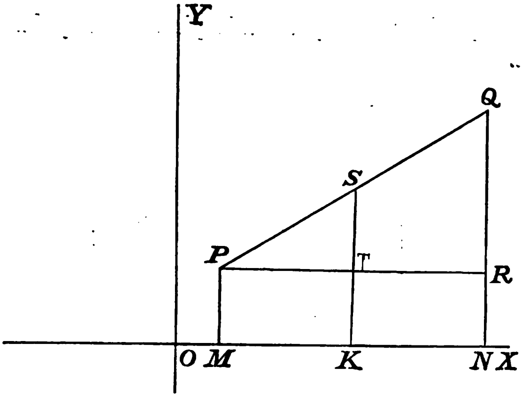


Let $PO = r_1$, $QO = r_2$, $POA = \theta_1$, $QOA = \theta_2$,

then $\delta^2 = PO^2 + OQ^2 - 2PO \,.\, OQ \cos POQ$

$= r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 \sim \theta_2)$.

**11.** *To find the coordinates of the point which bisects the straight line joining two given points.*



Let $S$ be the point required, $OK$, $SK$ its coordinates; let $SK$ cut $PR$ in $T$.

Then $OK = OM + MK = OM + PT = OM + \frac{1}{2} PR$

$$= x_1 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2}{2}.$$

Similarly $$SK = \frac{y_1 + y_2}{2}.$$

To find the coordinates of the point which divides the straight line joining two given points in a given ratio.

Let the given ratio be $m : n$.

Then, in the preceding figure,

$$PT : TR :: m : n;$$

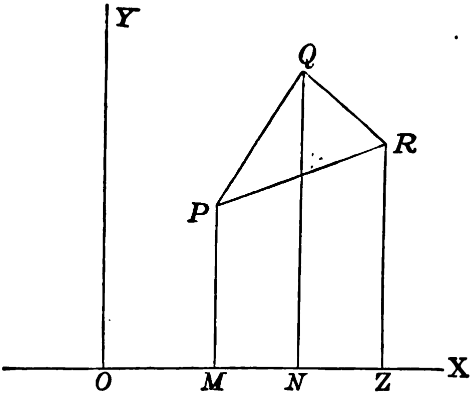
$$\therefore PT : PR :: m : m + n;$$

$$\therefore OK = x_1 + \frac{m}{m+n}(x_2 - x_1) = \frac{mx_2 + nx_1}{m+n}.$$

If $m + n = 1$, we have $OK = mx_2 + nx_1$, a form which is often useful.

Similarly $$SK = \frac{my_2 + ny_1}{m+n}.$$

**12.** *To find the area of the triangle whose angular points are given.*



Let $PQR$ be the triangle, and let the coordinates of $P, Q, R$ be $x_1y_1$, $x_2y_2$, $x_3y_3$ respectively.

Then area $PQR = PQNM + RLNQ - PRLM$.

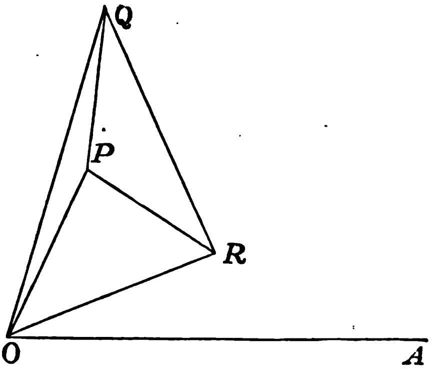
But $$PQNM = \tfrac{1}{2} MN (PM + QN) = \tfrac{1}{2} (x_2 - x_1)(y_2 + y_1).$$

Similarly, $$QRLM = \tfrac{1}{2}(x_3 - x_2)(y_3 + y_2),$$
$$RLMP = \tfrac{1}{2}(x_3 - x_1)(y_3 + y_1);$$

∴ area required

$$= \tfrac{1}{2}\{(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) - (x_3 - x_1)(y_3 + y_1)\}$$
$$= \tfrac{1}{2}\{x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1\}.$$

It is easy to see, that if the axes be inclined at an angle $\omega$, the preceding expression must be multiplied by $\sin \omega$.



If the coordinates be polar,

$$\text{let } OP = r_1, \quad OQ = r_2, \quad OR = r_3,$$

$$OPA = \theta_1, \; OQA = \theta_2, \; ORA = \theta_3,$$

then $$\Delta PQR = \Delta QOR - \Delta QOP - \Delta \overset{P}{\cancel{Q}}OR$$

$$= \tfrac{1}{2}\{r_2r_3 \sin(\theta_2 - \theta_3) - r_3r_1 \sin(\theta_1 - \theta_3) - r_1r_2 \sin(\theta_2 - \theta_1);$$

$$= \tfrac{1}{2}\{r_2r_3 \sin(\theta_2 - \theta_3) + r_3r_1 \sin(\theta_3 - \theta_1) + r_1r_2 \sin(\theta_1 - \theta_2)\}.$$

In these two expressions the coordinates must be taken in such an order as to make the whole expression positive.

## EXAMPLE I.

1. Indicate by a figure the relative position of the following points,

1, 1; 0, −3; 5, −4; 1, 0; −1, 0; −3, −2; −2, 6; 2, −6.

2. Indicate by a figure the relative position of the following points, whose coordinates are polar,

$$a,\ 15^\circ;\ 2a,\ 30^\circ;\ a\cos\alpha,\ \alpha;\ a\tan\alpha,\ \frac{\pi}{2}-\alpha;$$
$$-a,\ 45^\circ;\ -a,\ -135^\circ.$$

3. Shew that the coordinates $r,\ \theta;\ -r,\ \pi+\theta;\ -r,\ \theta-\pi$, represent the same point.

4. Determine the points whose coordinates satisfy the following equations:

(i) $x+y=3,\quad x-y=1.$

(ii) $3x+4y=7,\quad 4x-3y=2.$

(iii) $x+2y+3=0,\quad y=3x+4.$

(iv) $y^2=4ax,\quad x=y.$

(v) $x^2+y^2=c^2,\quad x+y=c.$

(vi) $x^2+y^2=5a^2,\quad x^2=4ay.$

5. Determine the distances of the following point from the origin, the axes being (i) rectangular, (ii) inclined at an angle 60°,

$$3a,\ 4a;\ -2b,\ b;\ a\sin\alpha,\ a\cos\alpha;\ a,\ -3a.$$

6. Determine the distances between the following pairs of points:

$$a,\ 0 \text{ and } 0,\ -a;\ h,\ k \text{ and } 2h,\ -3k;$$
$$a,\ b \text{ and } b,\ a;\ -3a,\ 2a \text{ and } -9a,\ -6a;$$
$$a\tan\alpha,\ b\sec\alpha \text{ and } a\cot\alpha,\ b\operatorname{cosec}\alpha;$$

the axes being rectangular.

7. The axes being inclined at an angle $\omega$, determine the distances between the following pairs of points,

$$a,\ 0 \text{ and } 0,\ a;\ 0,\ 0 \text{ and } a\cos\frac{\omega}{2},\ a\sin\frac{\omega}{2};$$

$$a\tan\frac{\omega}{2},\ -b\sec\frac{\omega}{2} \text{ and } a\cos\omega,\ b\sin\omega.$$

8. Determine the distances between the following points, whose coordinates are polar,

$$a,\ \theta \text{ and } b,\ \phi;\ a,\ \theta \text{ and } a,\ -\theta;\ a,\ \theta \text{ and } -a,\ -\theta;$$

$$2a,\ 30^\circ, \text{ and } a,\ 60^\circ.$$

9. Determine also the points of bisection and trisection of the lines joining the pairs of points in question 6.

10. Find the areas of the triangles whose angular points are

(i) $0,\ b;\ a,\ 0;\ a,\ b.$

(ii) $a,\ 2a;\ 2a,\ 3a;\ 3a,\ -4a.$

(iii) the origin, $x_1y_1$, $x_2y_2$.

(iv) the pole, $r,\ 0,\ r',\ \theta$

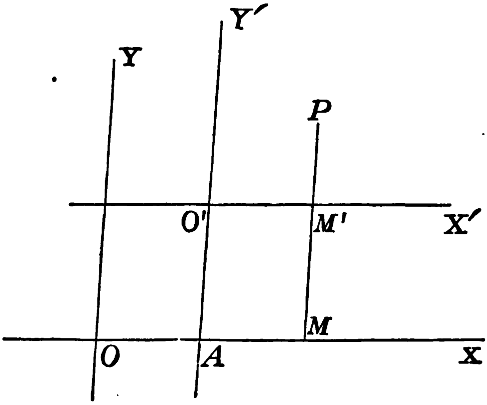
11. $ABC$ is a triangle, $D$, $E$ the centres of the circumscribing and inscribed circles: find the coordinates of $D$, $E$; (1) $B$ being the origin, $BC$ axis of $x$, and the axes rectangular; (2) $AB$, $AC$ being the axes; (3) $A$ being the pole, $AB$ the initial line.

12. In the triangle $ABC$, straight lines are drawn from the angles bisecting the sides; find the coordinates of their point of intersection, taking two of the sides as axes.

# CHAPTER II.

## TRANSFORMATION OF COORDINATES.

**13.** *To change the origin from one point to another, the direction of the axes remaining unaltered.*



Let $x$, $y$, $x'$, $y'$ be the coordinates of the *same* point $P$ referred to the old and new systems respectively: let $h$, $k$ be the coordinates of the new origin referred to the old axes.

Then $$AM = OA + AM = OA + OM',$$

or $$x = x' + h,$$

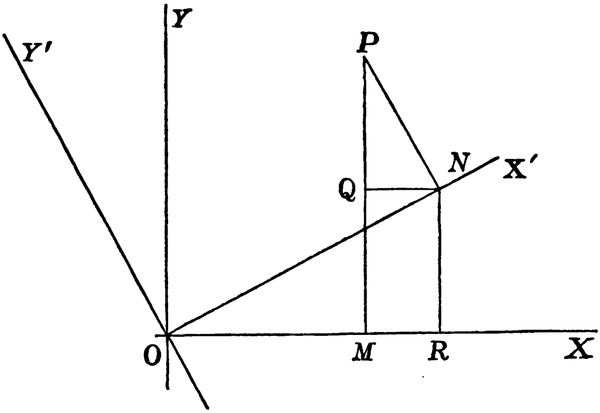
$$PM = MM' + PM' = O'A + PM',$$

$$y = y' + k.$$

Ex. Change the equation $x^2 + y^2 = a^2$, the origin being changed to a point $\alpha$, $\beta$.

Here
$$x = x' + \alpha,$$
$$y = y' + \beta;$$
$$\therefore\ (x' + \alpha)^2 + (y' + \beta)^2 = a^2,$$
the equation required.

**14.** *To change the coordinates from one rectangular system to another, the origin being unaltered.*



Let $OM = x$, $PM = y$, $ON = x'$, $PN = y'$,
$$XOX' = YOY' = \theta.$$

From $N$ draw $NQ$, $NR$ perpendicular to $PM$, $OX$, respectively;

then $OM = OR - RM = OR - NQ = ON\cos\theta - PN\sin\theta$,

or
$$x = x'\cos\theta - y'\sin\theta,$$
$$PM = MQ + QP = RN + QP = ON\sin\theta + PN\cos\theta,$$
or
$$y = x'\sin\theta + y'\cos\theta.$$

Ex. In the equation

$$x^2 - y^2 = a^2,$$

turn the axes through an angle $-45°$.

The equation becomes

$$\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 = a^2,$$

or $$2x'y' = a^2.$$

**15.** *To transform an equation from one rectangular system to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the original axes; next turn these axes through the required angle.

Thus, if $h$, $k$ be the coordinates of the new origin referred to the old axes, $\theta$ the angle between the original and final axes of $x$,

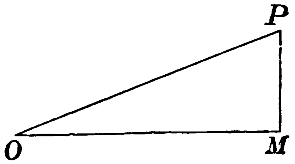
$$x = h + x' \cos\theta - y' \sin\theta,$$

$$y = k + x' \sin\theta + y' \cos\theta.$$

In all these transformations attention must be paid to the *sign* of $\theta$.

**16.** *To transform an equation from rectangular to polar coordinates.*

*a.* Let the origin of rectangular coordinates be the pole, and the axis of $x$ the initial line.



Then
$$OM = OP \cos POM,$$
$$\overset{P}{O}M = OP \sin POM,$$

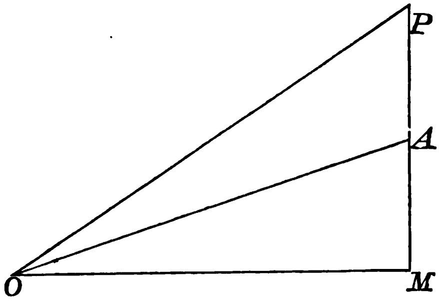
or
$$x = r\cos\theta,\quad y = r\sin\theta.$$

Conversely, if we wish to transform from polar to rectangular coordinates,

$$r^2 = r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2,$$

$$\tan\theta = \frac{y}{x}.$$

*β*. Let the origin be the pole, and let the initial line make an angle $\alpha$ with the axis of $x$.

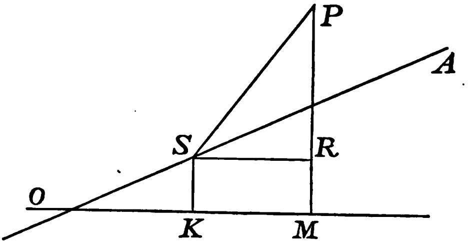


Then $OM = OP\cos POM = OP\cos(POA + AOM)$,

or
$$x = r\cos(\theta + \alpha).$$

Similarly
$$y = r\sin(\theta + \alpha).$$

$\gamma$. Next let the coordinates of the pole be $h$, $k$, and let the initial line make an angle $a$ with the axis of $x$.



Then $$OM = OK + KM = OK + SP \cos PSR,$$

or $$x = h + r\cos(\theta + a),$$

$$PM = SK + PR = SK + SP \sin PSR,$$

or $$y = k + r\sin(\theta + a).$$

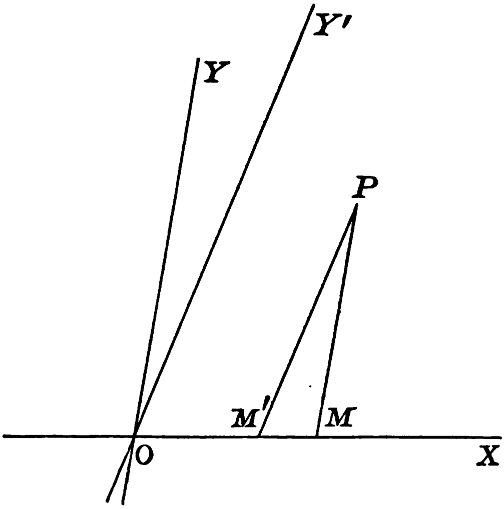
Conversely, $$r^2 = (x-h)^2 + (y-k)^2,$$

$$\tan(\theta + a) = \frac{y-k}{x-h}.$$

**17.** *To transform an equation from an oblique system to another, the origin and axis of $x$ remaining the same.*

Let $O$ be the origin, $OA$ the axis of $x$, $OY$ the old axis of $y$, $OY'$ the new axis of $y$; let $YOA = \omega$, $Y'OA = \omega'$.

Let $x$, $y$, $x'$, $y'$ be the coordinates of the same point $P$ referred to the old and new systems respectively.



Then $OM = OM' + M'M = OM' + PM' \dfrac{\sin M'PM}{\sin PMO}$,

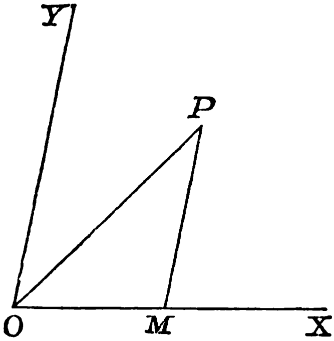
or
$$x = x' + y' \frac{\sin(\omega - \omega')}{\sin \omega},$$

$$PM = PM' \frac{\sin PM'A}{\sin PMA},$$

$$y = y' \frac{\sin \omega'}{\sin \omega}.$$

**18.** *To transform an equation from an oblique system to polar coordinates, the origin being the pole and the axis of $x$ the initial line.*

Let $OM = x$, $OM = y$, $OP = y$, $POA = \theta$.



Then $$OM = OP\frac{\sin MPO}{\sin PMA},$$

or $$x = r\frac{\sin(\omega - \theta)}{\sin \omega}.$$

Similarly, $$PM = OP\frac{\sin POM}{\sin PMA},$$

or $$y = r\frac{\sin \theta}{\sin \omega}.$$

We may write these values of $x$ and $y$ as

$$x = mr, \quad y = nr,$$

where $m$ and $n$ depend on the values of $\theta$ at the point.

If the coordinates of the pole be $h$, $k$, we shall have

$$x = h + mr, \quad y = k + nr.$$

Ex. 1. Transform the equation

$$x^2 + y^2 = c^2$$

to polar coordinates.

Here $$x = r\cos\theta, \quad y = r\sin\theta;$$

$$\therefore\ r^2(\cos^2\theta + \sin^2\theta) + a^2,$$

or $$r = a.$$

Ex. 2. Transform the equation

$$x \cos \alpha + y \sin \alpha = p$$

to polar coordinates, the coordinates of the pole being $p \cos \alpha$, $p \sin \alpha$, and the initial line being inclined to the axis of $x$ at an angle $\alpha$.

Here
$$x = p \cos \alpha + r \cos (\theta + \alpha),$$
$$y = p \sin \alpha + r \sin (\theta + \alpha);$$

$$\therefore\ p \cos^2 \alpha + r \cos (\theta + \alpha) \cos \alpha + p \sin^2 \theta + r \sin (\theta + \alpha) \sin \alpha = p,$$

or
$$r \cos \theta = 0, \quad \therefore\ \theta = \frac{\pi}{2}.$$

Ex. 3. Transform the equation

$$r = a \cos \theta$$

to rectangular coordinates, the origin coinciding with the pole, and the initial line with the axis of $x$.

Here
$$r^2 = ar \cos \theta,$$

or
$$x^2 + y^2 = ax.$$

**19.** General formulæ exist for transforming equations from a rectangular system to an oblique one, from an oblique to a rectangular, or from one rectangular system to another; but the student is advised to defer their consideration until he has become more familiar with the subject. Particular cases, in which such transformations are easily effected, will be found among the examples.

## EXAMPLE II.

1. If in the equation

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0,$$

the origin be changed to the point $(a, b)$, the equation becomes

$$x^2 + y^2 = c^2.$$

2. Transform the equation $x + y = c$ by turning the axes through an angle 45°.

3. Transform the equation

$$x^2 \cos^2 a - y^2 \sin^2 a = a^2$$

by turning the axes through an angle $a$.

4. Transform the equation $\frac{x}{a} + \frac{y}{b} = 1$ by transferring the origin to the point $\frac{a}{2}$, $\frac{b}{2}$, and turning the axes through an angle $\theta$ such that $\tan\theta = -\frac{b}{a}$.

5. Transform the equations

$$x \cos a + y \sin a = a,$$

$$x^2 + xy + y^2 = b^2,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

from rectangular to polar coordinates.

6. Transform the equation

$$y^2 = 4ax,$$

the coordinates of the pole being $a$, 0.

7. Transform the equation

$$r^2 = a^2 \cos 2\theta$$

from polar to rectangular coordinates.

8. If $x_1$, $y_1$, $x_2$, $y_2$, be the coordinates of two points $P$, $Q$, and $PQ$ make an angle $\theta$ with the axis of $x$,

$$x_2 = x_1 + r \cos\theta, \quad y_2 = y_1 + r \sin\theta,$$

where $PQ = r$.

9. If the new system have the same origin as the old, and the old axes bisect the angles between the new axes,

$$x = (x' + y') \cos\frac{\omega}{2}, \quad y = (y' - x') \sin\frac{\omega}{2}.$$

10. If both systems be rectangular and the equation to the old axis of $y$ be $x - y = 0$ referred to the new axes, the old and new axes are inclined to one another at an angle of 45°.

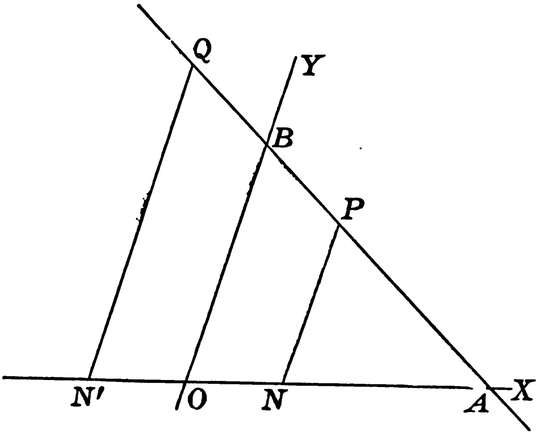
# CHAPTER III.

## THE STRAIGHT LINE.

### **20.** *Equation to a straight line.*

We have already seen that the equations $x = \text{const.}$, $y = \text{const.}$ represent straight lines parallel to the axes of $x$ and $y$ respectively.

We will now find the equation to a straight line in any position.



Let the straight line cut the axis of $x$ in the point $A$, and that of $y$ in the point $B$; let $OA = a$, $OB = b$; let the coordinates of $P$, any point in the line, be $x$ and $y$.

Then, $\because$ $PN$ is parallel to $BO$,

$$\therefore \frac{PN}{BO} = \frac{AN}{AO} = \frac{AO - ON}{AO},$$

or, $$\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a},$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1.$$

This equation is therefore satisfied by every point in $AB$, and by no other points, since it is deduced from the geometrical relation $PN : OB :: AN : OA$, which is only true when $P$ lies on $AB$.

It may easily be seen that this equation holds for all points on the line $AB$, or $AB$ produced either way.

For let $Q$ be such a point; then

$$\frac{QN'}{BO} = \frac{AN'}{AO} = 1 + \frac{ON'}{AO},$$

$$\therefore \frac{-ON'}{AO} + \frac{QN'}{BO} = 1,$$

but if $x$, $y$, be the coordinates of $Q$, $x = -ON'$, $y = QN'$,

$$\therefore \frac{x}{a} + \frac{y}{b} = 1, \text{ as before.}$$

Since this equation does not involve the angle $AOB$, it is true, whether the coordinates are rectangular or oblique.

Hence if we wish to find the equation to a straight line, we must find the distances from the origin of the points in which it cuts the axes, and denoting these distances by $a$ and $b$, the equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

will be the equation required.

In this equation multiply every term by any quantity $k$;

then
$$\frac{kx}{a}+\frac{ky}{b}=k.$$

Let
$$\frac{k}{a}=A,\ \frac{k}{b}=B,\ k=-C,$$

then the equation becomes
$$Ax+By+C=0,$$

the general equation of the first degree between the two variables $x$ and $y$.

**21.** *The equation* $Ax+By+C=0$ *always represents a straight line.*

We have proved that every straight line may be represented by an equation of the form
$$Ax+By+C=0,$$
we shall now prove that every equation of this form represents a straight line.

Let $P$ (in the preceding figure) be any point whose coordinates $x$ and $y$ satisfy the equation
$$Ax+By+C=0.$$

In this equation put $y=0$, $\therefore x=-\frac{C}{A}$;

again, put $x=0$, $\therefore y=-\frac{C}{B}$.

Let $OA=-\frac{C}{A}$, $OB=-\frac{C}{B}$, and join $AP$, $BP$.

Then $\frac{PN}{BO}=-\frac{By}{C}$, $\frac{AN}{AO}=\frac{AO-ON}{AO}=\frac{C+Ax}{C}$.

But $Ax + By + C = 0$, $\therefore \dfrac{-By}{C} = \dfrac{C + Ax}{C}$,

$$\therefore \frac{PN}{BO} = \frac{AN}{AO};$$

therefore the triangles $PAN$, $BAO$ are similar, and the angles $PAN$, $BAO$ equal, that is $P$ lies on the straight line joining $A$ and $B$, and the equation $Ax + By + C = 0$ represents that straight line.

The equation to a straight line may therefore be written as

$$Ax + By + C = 0.$$

This equation always represents *some* straight line, and may be made to represent *any* one by giving appropriate values to $A$, $B$, and $C$.

**22.** Since in this equation there are three constants $A$, $B$, $C$, it would seem that the equation might be made to satisfy *three* independent algebraical conditions: we know, however, that a straight line is completely determined if two points in it are known, and that a straight line can therefore be made to satisfy only *two* geometrical conditions.

The quantities $A$, $B$, $C$ are, however, not all really independent, for if we divide by any one of them, as $C$, the equation becomes

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0.$$

This equation is completely determined if we know the value of the ratios $\dfrac{A}{C}, \dfrac{B}{C}$; hence the equation to a straight line may be made to satisfy two algebraical conditions, which of course correspond to some two geometrical properties of the straight line represented.

It is important to notice that since $x$ and $y$ each represent a line, the ratios $\dfrac{C}{A}$, $\dfrac{C}{B}$, must each represent lines, that is, the quantities $A$ and $B$ must be each of one dimension in space less than $C$.

The equation

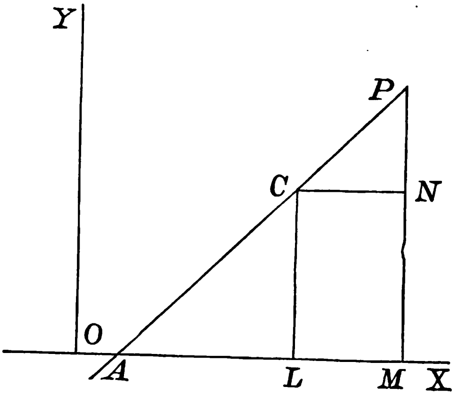
$$Ax+By+C=0$$

is called a linear equation in $x$ and $y$.

Let us now determine the various forms which this equation assumes according to the different conditions which the straight line is made to satisfy.

**23.** *To find the equation to a straight line which passes through a given point, and makes a given angle with the axis of $x$.*

Let $ACP$ be the straight line, $C$ the given point, whose coordinates are $x_1$ $y_1$, $CAx$ the given angle $= a$.



Then $$\tan a = \frac{PN}{CN} = \frac{y-y_1}{x-x_1},$$

or $$(y-y_1)\cos a = (x-x_1)\sin a.$$

If $\tan \alpha = m$, this equation may be put into the form

$$y - y_1 = m(x - x_1).$$

If the coordinates be oblique, $m$ is still equal to $\frac{PN}{CN}$,

$$m = \frac{CL}{AL} = \frac{\sin CAL}{\sin ACL} = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

which is constant since $\omega$ and $\alpha$ are constant.

Two particular cases are worthy of notice.

(1) Let the fixed point be the origin, then $x_1 = 0$, $y_1 = 0$.

The equation therefore becomes

$$y = mx.$$

(2) Let the fixed point be on the axis of $y$ at a distance $c$ from the origin, then the equation becomes

$$y - c = mx,$$

or

$$y = mx + c,$$

a form which, though unsymmetrical, is occasionally useful.

Comparing this equation with the form

$$Ax + By + C = 0,$$

we see that $m = -\frac{B}{A}$.

Hence if $\theta$ be the angle which the line $Ax + By + C = 0$ makes with the axis of $x$, $\tan \theta = -\frac{B}{A}$.

**24.** *To find the equation to the straight line which passes through two given points.*

Let the given points be $x_1y_1$, $x_2y_2$, and let

$$Ax + By + C = 0 \text{..............................} (1),$$

be the equation required.

Then since $(x_1\ y_1)$ $(x_2\ y_2)$ lie on (1),

$$Ax_1 + By_1 + C = 0 \text{..........................} (2),$$

$$Ax_2 + By_2 + C = 0 \text{..........................} (3)$$

Eliminate $A$, $B$, and $C$, thus:

Subtract (2) from (1),

then $$A(x - x_1) + B(y - y_1) = 0.$$

Subtract (3) from (1), 2

then $$A(x_2 - x_1) + B(y_2 - y_1) = 0;$$

$$\therefore \quad \frac{A}{B} = -\frac{y_2 - y_1}{x_2 - x_1};$$

$$\therefore \quad \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

the equation required.

Let one of the points $(x_1\ y_1)$ be the origin, then $x_1 = 0$, $y_1 = 0$, and the equation becomes

$$\frac{y}{y_2} = \frac{x}{x_2}.$$

The equation $A(x - x_1) + B(y - y_1) = 0$, $\therefore$ represents any straight line through the point $x_1$, $y_1$.

**25.** *To find the equation to a straight line which is parallel to a given straight line.*

Let $Ax + By + C = 0$, be the equation to the given straight line,

$A'x + B'y + C' = 0$, to the required straight line.

Then since they are parallel they make the same angle with the axis of $x$,

$$\therefore \frac{B'}{A'} = \frac{B}{A}.$$

The equation to the required line is therefore

$$A'x + \frac{B}{A} A'y + C' = 0,$$

or
$$Ax + By + \frac{C'A}{A'} = 0,$$

where $\frac{C'}{A'} A$ is indeterminate, as we might have conjectured, since there are an infinite number of straight lines which are parallel to a given straight line.

If the line also pass through the point $(x_1\ y_1)$

$$Ax_1 + By_1 + C_1 = 0,$$

and therefore
$$A(x - x_1) + B(y - y_1) = 0.$$

If the equation to the given line be

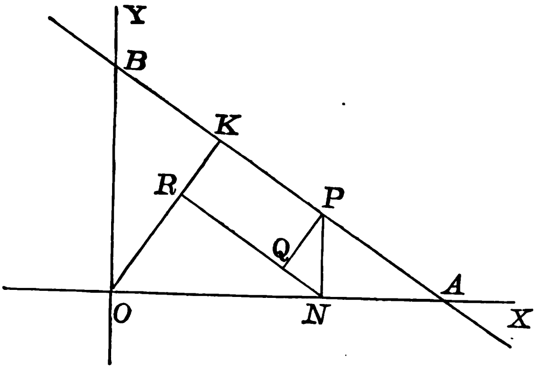
$$\frac{x}{a} + \frac{y}{b} = 1,$$

that to any straight line parallel to it will be

$$\frac{x}{a} + \frac{y}{b} = \lambda,$$

where $\lambda$ may have any assigned value.

**26.** *To find the equation to a straight line in terms of the perpendicular from the origin, and the angle which this perpendicular makes with the axis of* x.



Let $OK = p$, $KOA = a$.

Draw $NR$ parallel to $AB$, cutting $OK$ in $R$, and $PQ$ perpendicular to $NR$.

Then $OR + RK = OK = p$,

but $OR = ON \cos a = x \cos a$,

$$RK = PQ = PN \sin a = y \sin a;$$

$$\therefore\ x \cos a + y \sin a = p,$$

the equation required.

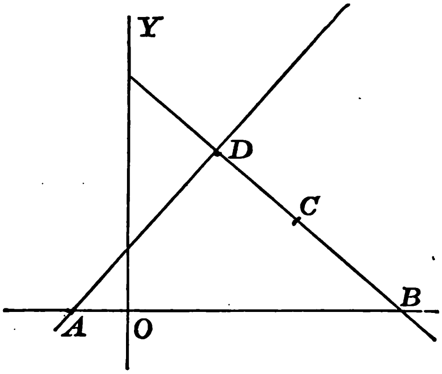
If the axes be oblique, let $KOB = \beta$,

then $PNQ = \beta$, $PQ = y \cos \beta$,

and the equation becomes

$$x \cos a + y \cos \beta = p.$$

**27.** *To find the equation to the straight line which passes through a given point, and is perpendicular to a given straight line.*



Let $C$ be the given point whose coordinates are $h$, $k$, $AD$ the given straight line, whose equation is

$$Ax + By + C = 0.$$

Let $BCD$ be the line required,

then $$y - k = (x - h) \tan DBX$$

is the equation required, where $\tan DBX$ has to be determined.

Now $\tan DBX = -\tan DBA = -\cot DAB = \frac{B}{A}$;

$$\therefore\ y - k = \frac{B}{A}(x - h),$$

or $$A(y - k) = B(x - h)$$

is the equation required.

If the original equation to the straight line were

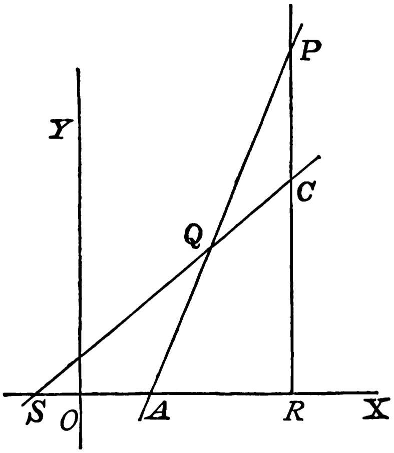
$$y = mx + c, \text{ or } \frac{x}{a} + \frac{y}{b} = 1,$$

the required equations would become

$$y - k + \frac{1}{m}(x - h) = 0,$$

or $$\frac{y-k}{a} = \frac{x-h}{b} \text{ respectively.}$$

**28.** *To find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.*



Let $C$ be the given point whose coordinates are $x_1$ $y_1$, $AQP$ the given straight line whose equation is

$$Ax + By + C = 0.$$

Let $PCR$, $CQS$ be the straight lines required, making the equal angles $CPQ$, $CQP$ with $AQP$.

Let $$CPQ = CQP = \alpha,$$

then the equation to $RCP$ is

$$y - y_1 = (x - x_1) \tan PRX;$$

But $\tan PRX = \tan(PAx + \alpha)$

$$= \frac{\tan PAx + \tan\alpha}{1 - \tan PAx \tan\alpha}$$

$$= \frac{-\frac{B}{A} + \tan\alpha}{1 + \frac{B}{A}\tan\alpha} = \frac{A\sin\alpha - B\cos\alpha}{A\cos\alpha + B\sin\alpha};$$

$$\therefore\ (A\cos\alpha + B\sin\alpha)(y - y_1) = (A\sin\alpha - B\cos\alpha)(x - x_1)$$

is the equation to $CP$.

Similarly the equation to $CQ$ is

$$(A\cos\alpha - B\sin\alpha)(y - y_1) + (A\sin\alpha + B\cos\alpha)(x - x_1) = 0.$$

If $PAX = \beta$ these equations evidently become

$$y - y_1 = \tan(\beta \pm \alpha)(x - x_1).$$

If the straight lines are to be perpendicular to the given straight line, $P$ and $Q$ coincide, and the equations become

$$(x - x_1)\cos\alpha + (y - y_1)\sin\alpha = 0.$$

**29.** *To find the polar equation to a straight line.*

Let the equation to the straight line referred to rectangular axes in which the origin coincides with the pole and the axis of $x$ with the initial line be

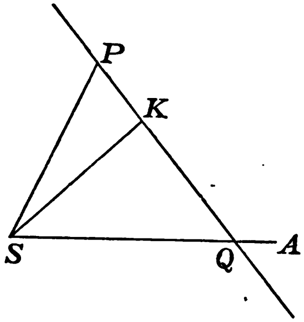
$$Ax + By + C = 0.$$

Then $$x = r\cos\theta,\ y = r\sin\theta;$$

$$\therefore\ Ar\cos\theta + Br\sin\theta + C = 0,$$

the polar equation required.

**30.** Let us investigate the polar equation to a straight line independently.



Let $SK$ be drawn from $S$ perpendicular to the straight line $PQ$, let $SK=p$, $KSQ=\alpha$; let the coordinates of any point $P$ be $r$, $\theta$.

Then $$SP=SK\sec PSK,$$

or $$r=p\sec(\theta-\alpha),$$

$$r\cos(\theta-\alpha)=p.$$

If in this equation we expand $\cos(\theta-\alpha)$ we obtain

$$r\cos\theta\cos\alpha+r\sin\theta\sin\alpha=p.$$

Since this equation and

$$Ar\cos\theta+Br\sin\theta+C=0,$$

both represent the same straight line, we must have

$$\frac{\cos\alpha}{A}=\frac{\sin\alpha}{B}=\frac{-p}{C}.$$

**31.** Let us collect and compare the different forms of the equation to a straight line referred to rectangular axes.

We have first the general form

$$Ax+By+C=0 \text{ ........................ } (1).$$

Next, if $a$, $b$, be the intercepts on the axes

$$\frac{x}{a}+\frac{y}{b}=1 \text{ ........................ } (2).$$

Next, if $c$ be the intercept on the axis of $y$, $\theta$ the ∠ which the straight line makes with the axis of $x$,

$$y = x \tan\theta + c \text{ ......................... } (3).$$

Next, if the line passes through $(x_1\, y_1)$ $(x_2\, y_2)$,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \text{ ..................... } (4).$$

Next, if $p$ be perpendicular from the origin, $a$ the angle which it makes with the axis of $x$,

$$x \cos a + y \sin a = p \text{ ...................... } (5).$$

In (1) if $A = 0$, $y = -\dfrac{C}{A}$, or the straight line is parallel to the axis of $x$.

If $B = 0$, $x = -\dfrac{C}{A}$, or the line is parallel to that of $x$.

If $C = 0$, it passes through the origin.

If $A = 0$, $C = 0$, $y = 0$, that is the line becomes the axis of $x$.

If $B = 0$, $C = 0$, it becomes the axis of $y$.

Next, to compare (1) and (2), divide (1) by $C$;

$$\therefore -\frac{Ax}{C} - \frac{By}{C} = 1, \quad \frac{x}{a} + \frac{y}{b} = 1,$$

must be identical equations,

$$\therefore -\frac{C}{A} = a, \quad -\frac{C}{B} = b.$$

If we compare (1) and (3) we get

$$-\frac{C}{B} = c, \quad -\frac{A}{B} = \tan\theta,$$

$$\therefore \cos\theta = \frac{-B}{(A^2 + B^2)^{\frac{1}{2}}}, \quad \sin\theta = \frac{A}{(A^2 + B^2)^{\frac{1}{2}}}.$$

The angle $\theta$ being considered to be less than two right angles, and the sine therefore positive.

Again, to compare (1) and (4), divide (1) by $(A^2+B^2)^{\frac{1}{2}}$.

Hence we obtain

$$\frac{Ax}{(A^2+B^2)^{\frac{1}{2}}}+\frac{By}{(A^2+B^2)^{\frac{1}{2}}}=\frac{-C}{(A^2+B^2)^{\frac{1}{2}}}.$$

Here we may put

$$\frac{A}{(A^2+B^2)^{\frac{1}{2}}}=\cos\alpha,\quad \frac{B}{(A^2+B^2)^{\frac{1}{2}}}=\sin\alpha,$$

and

$$\therefore \frac{-C}{(A^2+B^2)^{\frac{1}{2}}}=p.$$

Hence the distance from the origin of any straight line whose equation is

$$Ax+By+C=0,$$

is

$$\frac{-C}{(A^2+B^2)^{\frac{1}{2}}}.$$

Since this distance is essentially positive, we must take that value of $(A^2+B^2)^{\frac{1}{2}}$ which is of a different sign from $C$.

Again, if we compare equations (2) and (3),

$$b=c,\quad \tan\theta=-\frac{b}{a}.$$

The truth of these equations is at once evident from geometrical considerations.

The student will find it a useful exercise to compare the remaining equations, and to obtain the other relations between the various constants involved in them.

We may remark that if, in the equation to a straight line, the coefficients of $x$ and $y$ have the same sign, the line makes an obtuse angle with the axis of $x$; if they are of different signs, an acute.

**32.** Although the equation to a straight line is of one dimension only, yet it does not follow that every equation of more than one dimension does not represent straight lines.

Take for instance the equation

$$xy - a(x+y) + a^2 = 0.$$

This may be put in the form $(x-a)(y-a)=0$,

$$\therefore \text{ either } x - a = 0, \text{ or } y - a = 0,$$

that is to say, the locus of the equation is two straight lines. Generally, whenever an equation can be resolved into simple factors it is satisfied by putting each separate factor equal to zero, and therefore is the locus of the various straight lines whose equations are so obtained.

The locus of an equation may be a point, or may be impossible. For instance,

$$(x-y)^2 + (x+y+a)^2 = 0.$$

Here $\qquad x - y = 0$, and also $x + y + a = 0$;

$$\therefore x = y = -\frac{a}{2}$$

is the only point which satisfies the equation.

This equation is really equivalent to two separate equations, since the sum of two squares can only vanish when each square vanishes.

Similarly, if we have the equation

$$x^2 + (x-y)^2 + a^2 = 0,$$

the locus is impossible, since no real values of $x$ and $y$ can be found such that $x^2 + (x-y)^2$ shall be negative.

### EXAMPLES.

1. Find the equation to the straight line which passes through the points $(a, b)$, $(b, a)$.

The equation required is

$$\frac{y-b}{a-b}=\frac{x-a}{b-a},$$

or $$x+y=a+b.$$

2. Find the equation to the straight line which passes through the point $(hk)$ and makes an angle $\frac{\pi}{3}$ with the axis of $x$.

The equation is evidently

$$y-k=\sqrt{3}\,(x-h).$$

If we wish to determine the points in which this line cuts the axis, put $$y=0;\ \therefore\ x=h-\frac{k}{\sqrt{3}}.$$

Again, put $$x=0;\ \therefore\ y=k-\sqrt{3}h.$$

Find the equation to the straight line through the origin perpendicular to

$$Ax+By+C=0.$$

Let $$A'x+B'y=0$$

be the equation required.

Then, since these lines are at right angles,

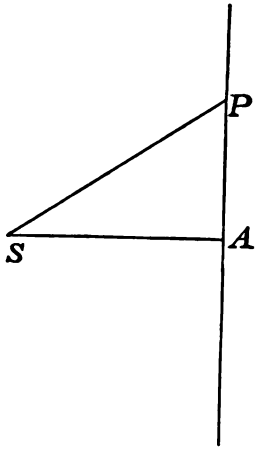
$$\frac{A'}{B'}=-\frac{B}{A};$$

$\therefore\ Bx=Ay$ is the equation required.

3. Find the polar equation to the straight line which cuts the initial line at right angles at a distance $a$ from the pole.

Let $PA$ be the line required. Then $SP\cos PSA=SA$,

or $$r\cos\theta=a.$$



Similarly the equation to a straight line parallel to the initial line, at a distance $b$ from it will be

$$r\sin\theta = b.$$

4. Draw the straight lines corresponding to the following equations:

$$3x + 4y = 12 \text{ ... .......................... } (1).$$

$$\frac{y}{3} = \frac{x-2}{4} \text{ .............................. } (2).$$

$$x + 2y + 3 = 0 \text{ ......................... } (3).$$

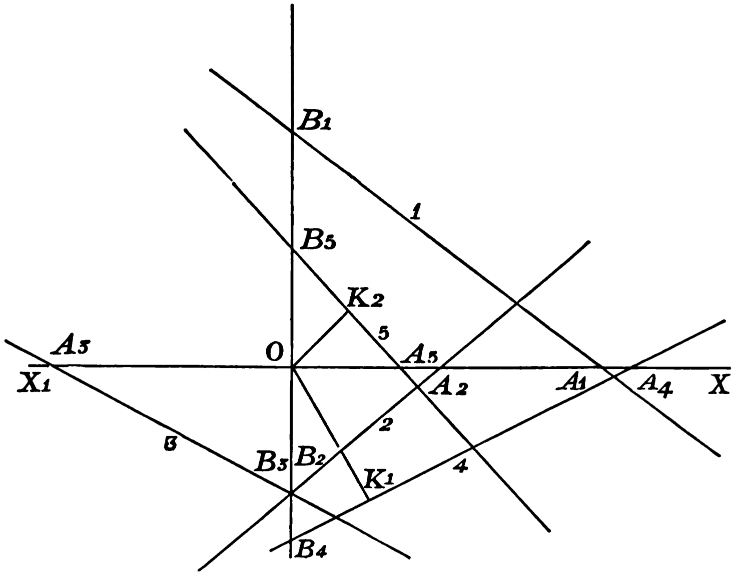
$$x\cos\frac{\pi}{3} - y\sin\frac{\pi}{3} = 2 \text{.................. } (4).$$

$$r(\cos\theta + \sin\theta) = \sqrt{2} \text{.................. } (5).$$

Equation (1) may be put in the form

$$\frac{x}{4} + \frac{y}{3} = 1.$$

The line represented cuts the axes in points distant 4 and 3 respectively from the origin.



Hence draw $OA_1 = 4$,

$OB_1 = 3$;

∴ $A_1B_1$ is the straight line required.

Similarly in equation (2) put $y = 0$;

$$\therefore x = 2.$$

Put $x = 0, \ \therefore y = -\frac{3}{2}$,

∴ measure $OA_2 = 2, OB_2 = -\frac{3}{2}$, $A_2B_2$ is the straight line required.

$B_2$ is taken on the negative part of the axis of $y$ because when $x$ vanishes in equation (2) $y$ is negative.

Similarly in equation (3) $OA_3 = -3$, $OB_3 = -\frac{3}{2}$, $A_3B_3$ is the line required.

Equation (4) may be put into the form

$$x\cos\left(-\frac{\pi}{3}\right)+y\sin\left(-\frac{\pi}{3}\right)=2.$$

$A_4K_1$ is therefore the straight line required, where $OK_1=2$, $A_4OK_1=\frac{\pi}{3}$, and being drawn in the negative direction is negative.

Equation (5) may be written

$$r\cos\left(\theta-\frac{\pi}{4}\right)=1.$$

Comparing this with the form $r\cos(\theta-a)=p$, we see that it represents the straight line $A_5B_5$ such that $OK_2=1$,

$$K_2OA_5=\frac{\pi}{4}.$$

## EXAMPLES III.

N.B. In the answer to each question a figure should be drawn.

Draw the straight lines represented by the following equations:

1. $x=y$. 2. $x+y=0$. 3. $mx+ny=0$.

4. $\frac{x}{2}+\frac{y}{3}=1$. 5. $\frac{x}{a}+\frac{y}{b}=3$. 6. $\frac{x-h}{l}=\frac{y-k}{m}$.

7. $y=\sqrt{3}x+1$. 8. $x\sin\theta-y\cos\theta=p$. 9. $y=3(x-2)$.

10. $x\cos\theta+y\sin\theta=a$. 11. $x\cos\theta-y\sin\theta=a\cos\theta$.

12. $x-y=a-b$. 13. $x+2y+3a=0$.

14. $5x-2y=a$. 15. $x+y+a=0$. 16. $\theta=a$.

17. $r\cos(\theta-a)=p$. 18. $r\cos\theta+5r\sin\theta=6a$.

19. If the axes be inclined at an angle $\omega$, find the equations to the straight lines which pass through a given point on the axis of $x$, and are perpendicular to the axes.

20. Find the equation to the straight line which passes through the point $(a, b)$, and makes an angle of 30° with the axis of $x$.

21. Find the equations to the straight lines which pass through the following pairs of points:

$a, b;\ b, a:\ h, k;\ -h, -k:\ h, k;\ h, -k:\ h, -k;\ -h -k:$ $a\cos\theta,\ a\sin\theta;\ b\cos\phi,\ b\sin\phi:\ 3,\ 4;\ 1,\ 2:\ 5,\ 6;\ -1,\ 0;$ $1, 2;\ 2, 4.$

22. Find the equations to the straight lines which are parallel to $\frac{x}{a}+\frac{y}{b}=1$, and pass respectively through the points

$$10,\ 0;\ 0,\ 2b;\ 3a,\ 0;\ h,\ k;\ a,\ b:\ -a,\ b.$$

23. Find the equation to the straight line which is parallel to $Ax+By+C=0$, and cuts off a given abscissa $a$ from the axis of $x$.

24. Find the equation to the straight line which is parallel to $\frac{x}{a}+\frac{y}{b}=1$, and cuts off a given triangle between itself and the axes.

25. Find the equation to the straight line which is parallel to $y=mx+b$, and at a distance $a$ from the origin.

26. Find the equation to the straight line which is at a distance $a\cos\alpha$ from the origin and makes an angle $-\alpha$ with the axis of $x$.

27. Find the equation to the straight line when the perpendicular from the origin makes an angle $-\alpha$ with the axis of $x$, and its length is $b$.

28. Find the perpendiculars from the origin on the following lines:

$$\frac{x}{a}+\frac{y}{b}=2;\qquad y-b=m(x-a);$$

$$3x+4y=7;\quad x-y=1;\quad x+3y+4=0.$$

29. Find the equation to the straight line which is perpendicular to $Ax + By + C = 0$, and cuts off a given length $b$ from the axis of $y$.

30. The equation to the straight line which is perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$, and which passes through the point $a$, $b$, is

$$ax - by = a^2 - b^2.$$

31. Determine the equation to the straight line which is perpendicular to $y = mx + c$, and cuts off a given area from the axes.

32. Find the equations to the straight lines which pass through the point $(h, k)$, and make an angle $\alpha$ with $\frac{x}{a} + \frac{y}{b} = 1$.

33. Also which pass through a given point on the axis of $y$, and make an angle $\beta$ with $\cos\alpha + y \sin\alpha = p$.

34. Determine the distances from the origin of the 2 lines in (33).

If the axes be inclined at an angle $\omega$,

35. Prove that if $\theta$ be the angle which the line

$$y = mx + c$$

makes with the axes of $x$, $m = \frac{\sin\theta}{\sin(\omega - \theta)}$;

36. Prove that if the straight lines

$$A_1x + B_1y + C_1 = 0, \qquad A_2x + B_2y + C_2 = 0,$$

are at right angles,

$$A_1A_2 - B_1B_2 + (A_1B_2 + A_2B_1)\cos\omega = 0.$$

37. Find the equations to two straight lines which pass through a given point on the axis of $y$, and make equal angles with that axis.

38. Find the equations to two straight lines which pass through a given point on the axis of $y$, and make equal angles with the axis of $x$.

39. Find the polar equations to the straight lines which pass through the following pairs of points:

$$r_1\,\theta_1;\ r_2\,\theta_2:\ a,\ 0;\ b,\ \frac{\pi}{2}:\ -a,\ 30^\circ;\ a,\ 60^\circ:\ a,\ 90^\circ;\ 6,\ 0.$$

40. If the axis of $x$ be the initial line, and the axis of $y$ make with it an angle $\omega$, shew that an equation referred to oblique coordinates may be transformed to polar by substituting $mr$, $nr$ for $x$, $y$, respectively, and find the values of $m$ and $n$.

41. Prove that the following equations represent two or more straight lines:

$$xy = 0,\ x^2 - y^2 = 0,$$

$$AB(x^2 + y^2) + (A^2 + B^2)xy + Bx + Ay = 0,$$

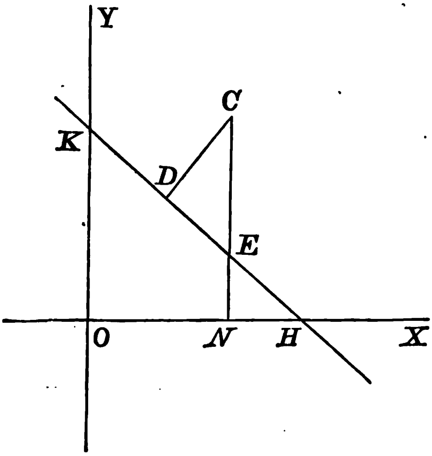
$$x^2 \cos^2\theta - y^2 \sin^2\theta = p^2 - 2py \sin\theta.$$

# CHAPTER IV.

## PROBLEMS ON THE STRAIGHT LINE.

WE may apply the equations which we have already found to the solution of various problems.

**33.** *Find the distance of a given point from a given straight line.*



Let the coordinates of the point be $x_1$, $y_1$, and the equation to the given straight line

$$Ax + By + C = 0,$$

Let $C$ be the given point, $HK$ the straight line; draw $CD$ perpendicular to $HK$, and let $EN$ the ordinate of $C$ cut $HK$ in $E$.

Then $$CD = CE \sin CED = CE \cos KHO$$
$$= - CE \cos KHX.$$

But $$CE = CN - EN = y_1 - EN,$$

and $E$ is a point on the line $Ax + By + C = 0$;

$$\therefore \quad B \,.\, EN = -C - Ax_1, \text{ since } ON = x_1.$$

Also $$\tan KHX = -\frac{A}{B};$$

$$\therefore \quad \cos KHX = -\frac{B}{(A^2 + B^2)^{\frac{1}{2}}};$$

$$\therefore \quad CD = \frac{-Ax_1 - By_1 - C}{(A^2 + B^2)^{\frac{1}{2}}},$$

or, since the root of $(A^2 + B^2)^{\frac{1}{2}}$ is indeterminate,

$$CD = \pm \frac{Ax_1 + By_1 + C}{(A^2 + B^2)^{\frac{1}{2}}}.$$

We shall see the meaning of the double sign if we put the equation to the straight line in the form

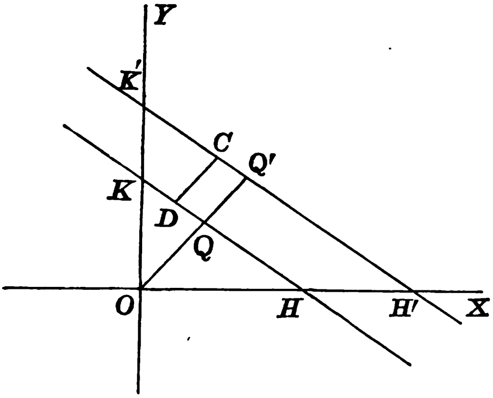
$$x \cos a + y \sin a = p \text{..........................} (2).$$

Through the point $x_1 \, y_1$ draw a straight line parallel to (2).

Its equation is evidently

$$x \cos a + y \sin a = p',$$

where $p'$ or $OQ'$ is the perpendicular from the origin $O$.



The distance between these lines is evidently the difference of their distances from the origin, i. e. $CD$, $QQ'$, $Q'O - QO$, or $p' - p$.

But
$$x_1 \cos \alpha + y_1 \cos \alpha = p';$$
therefore the distance required
$$= x_1 \cos \alpha + y_1 \sin \alpha - p.$$

This will be positive if $C$ lies as in the figure, for then $OQ'$, i. e. $p'$, or $x_1 \cos \alpha + y_1 \sin \alpha$, is greater than $p$, negative if $C$ lies between the origin and $HK$.

Hence we may consider the perpendicular from a point on a straight line positive if drawn in the *positive* direction *from* the line, negative if in the negative direction.

If equations (1) and (2) represent the same line, we have
$$\frac{A}{(A^2+B^2)^{\frac{1}{2}}} = \cos \alpha, \quad \frac{B}{(A^2+B^2)^{\frac{1}{2}}} = \sin \alpha, \quad -\frac{C}{(A^2+B^2)^{\frac{1}{2}}} = p,$$
and the expressions for the perpendiculars coincide.

Hence we see that, to get the perpendicular from a point on a line, we have only to substitute the coordinates of the point in the equation to the line, and divide by the square root of the sum of the squares of the coefficients of $x$ and $y$.

Thus the distance of the point $a$, $b$, from the line

$$\frac{x}{a}+\frac{y}{b}=1, \text{ or } \frac{x}{a}+\frac{y}{b}-1=0$$

is

$$\frac{\frac{a}{a}+\frac{b}{b}-1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{\frac{1}{2}}}, \text{ or } \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}.$$

The distance of the origin from the straight line

$$x\cos\alpha+y\sin\alpha=p$$

is $p$, and that therefore from

$$Ax+By+C=0$$

must be

$$\frac{\pm C}{(A^2+B^2)^{\frac{1}{2}}}.$$

**34.** *To find the distance between two parallel straight lines.*

Let their equations be

$$Ax+By+C=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$Ax+By+C'=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Take any point $x_1$, $y_1$ *between* them; then the distance of $x_1$, $y_1$ from (1) is

$$\frac{Ax_1+By_1+C}{(A^2+B^2)^{\frac{1}{2}}},$$

from (2) measured in the *opposite* direction

$$\frac{-Ax_1-By_1-C'}{(A^2+B^2)^{\frac{1}{2}}};$$

therefore the distance between (1) and (2) is the sum of these distances, or

$$\frac{C - C'}{(A^2 + B^2)^{\frac{1}{2}}}.$$

**35.** *To find the angle between two straight lines.*

Let their equations be

$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 = 0.$$

Then, if $\theta_1$, $\theta_2$ be the angles they make with the axis of $x$,

$$\tan\theta_1 = -\frac{A_1}{B_1},\ \tan\theta_2 = -\frac{A_2}{B_2};$$

$$\therefore\ \tan(\theta_1 - \theta_2) = \frac{\frac{A_2}{B_2} - \frac{A_1}{B_1}}{1 + \frac{A_1A_2}{B_1B_2}} = \frac{B_1A_2 - B_2A_1}{A_1A_2 + B_1B_2}.$$

These straight lines will evidently be parallel if $\theta_1 - \theta_2 = 0$, or

$$\frac{A_1}{B_1} = \frac{A_2}{B_2}.$$

They will be at right angles if $\theta_1 - \theta_2 = \frac{\pi}{2}$, or if $\tan(\theta_1 - \theta_2) = \infty$, or if

$$1 + \frac{A_1A_2}{B_1B_2} = 0.$$

**36.** *To find the coordinates of the point of intersection of two straight lines.*

Since the point lies on each of the straight lines, its coordinates must satisfy each of their equations.

Therefore, taking the same equations as before, multiplying the first by $B_2$, the second by $B_1$, and subtracting, we eliminate $y$, and obtain

$$(A_1B_2 - A_2B_1)\,x + (C_1B_2 - C_2B_1) = 0,$$

$$x = -\frac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1}.$$

By the same process, or *by symmetry*,

$$y = -\frac{C_1A_2 - C_2A_1}{B_1A_2 - B_2A_1}.$$

It may be observed that we have here written down the value of $y$ from that of $x$ without going through the process; we are able to do this because $x$ and $y$ are symmetrically involved in these equations; for wherever $x$ occurs with $A_1$ or $A_2$ as its factor $y$ occurs also with $B_1$ or $B_2$ as the corresponding factor, and therefore if in any results we substitute $B_1$ for $A_1$, $B_2$ for $A_2$, we shall get $y$ instead of $x$.

It will be also observed that the values of $x$ and $y$ obtained from these equations have for their denominators quantities of the same value but of opposite sign.

**37.** *To find the condition that three points may be on the same straight line.*

Let $x_1\,y_1$, $x_2\,y_2$, $x_3\,y_3$, be the coordinates of the points; then, since these points lie on a straight line, the area of the triangle of which these are the angular points must be equal to 0, but twice the area of this triangle is equal to

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3;$$

$$\therefore\ x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$$

is the condition required.

Generally however the best way of proving that three points lie on a straight line, is to find the equations to the straight lines passing through them taken two and two; if these equations are identical, the three points must lie on the line represented by either of them.

**38.** *To find the equation of a straight line which passes through the point of intersection of two given straight lines.*

Let the equations to the two straight lines be

$$A_1x + B_1y + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$A_2x + B_2y + C_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiply equation (2) by any quantity $\lambda$ and add it to (1), then $$(A_1 + \lambda A_2)x + (B_1 + \lambda B_2)y + C_1 + \lambda C_2 = 0 \ldots\ldots(3).$$

This is evidently the equation to some straight line, since it is of the first degree; also it passes through the intersection of (1) and (2), because if one of these equations is satisfied the other must be; it is therefore the equation required. The quantity $\lambda$ is called an indeterminate multiplier, its value depends on the different geometrical circumstances of each particular case.

For instance, let us find the equation to the line bisecting the angle between (1) and (2).

Let $\delta_1$, $\delta_2$ be the perpendiculars from any point of this line on the lines (1), (2), respectively, then since equation (3) may be written

$$\delta_1 (A_1^2 + B_1^2)^{\frac{1}{2}} + \lambda\delta_2(A_2^2 + B_2^2)^{\frac{1}{2}} = 0.$$

$$\delta_1 = \frac{A_1x + B_1y + C}{(A_1^2 + B_1^2)^{\frac{1}{2}}},$$

$$\delta_2 = -\frac{A_2x + B_2y + C_2}{(A_2^2 + B_2^2)^{\frac{1}{2}}},$$

Therefore since $\delta_1$, $\delta_2$ are proportional to the sines of the angles which (3) make with (1), (2), respectively, $\delta_1 = \delta_2$.

$$\therefore\ \lambda = -\frac{(A_1^2 + B_1^2)^{\frac{1}{2}}}{(A_2^2 + B_2^2)^{\frac{1}{2}}},$$

and the equation may be written

$$\frac{A_1x+B_1y+C}{(A_1^2+B_1^2)^{\frac{1}{2}}}=\frac{A_2x+B_2y+C}{(A_2^2+B_2^2)^{\frac{1}{2}}}.$$

$\delta_1$, $\delta_2$, must be of different signs, since they are measured in different directions from the line (3).

**39.** *Similarly to find the condition that three straight lines should pass through the same point.*

Let the equation to the third line be

$$A_3x+B_3y+C_3=0.$$

Then, since this straight line passes through the intersection of

$$A_1x+B_1y+C=0,\quad A_2x+B_2y+C_2=0,$$

its equation must be of the form

$$(A_1+\lambda A_2)x+(B_1+\lambda B_2)\,y+C_1+\lambda C_2=0\,;$$

$$\therefore\ \frac{A_1+\lambda A_2}{A_3}=\frac{B_1+\lambda B_2}{B_3}=\frac{C_1+\lambda C_2}{C_3}\,;$$

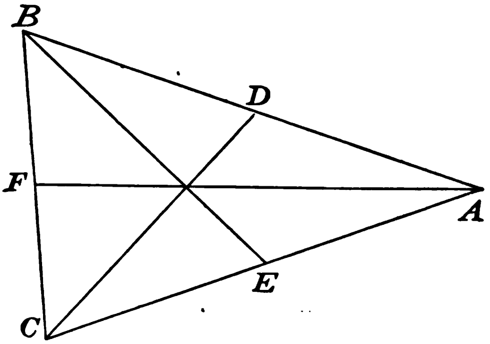
$$\therefore\ \lambda=\frac{B_1C_3-B_3C_1}{B_2C_3-B_3C_2}=\frac{A_1C_3-A_3C_1}{A_2C_3-A_3C_2},$$

the condition required.

**40.** Generally, however, problems of this kind may be solved more easily from the particular conditions of the problem.

Ex. Prove that the straight lines through the angular points of a triangle bisecting the opposite sides meet in a point.

Take the two sides $CA$, $CB$ as coordinate axes, let $D$, $E$, $F$ be the bisections of the sides.



The coordinates of $D$ are $\frac{a}{2}, \frac{b}{2}$;

$$\therefore \text{ the equation to } CD \text{ is } \frac{x}{a} = \frac{y}{b}.$$

The equations to $EB$, $AF$ are evidently

$$\frac{2x}{a} + \frac{y}{b} = 1,$$

$$\frac{x}{a} + \frac{2y}{b} = 1;$$

therefore the equation to a straight line through their intersection is

$$\frac{2+\lambda}{a} x + \frac{1+2\lambda}{b} y = 1 + \lambda.$$

Let $\lambda = -1$;

$$\therefore \frac{x}{a} = \frac{y}{b},$$

the equation to $CD$ already found.

## EXAMPLE IV.

1. Find the distance of the point 2, 3 from the line $x+y=1$.

2. Find the distance of the point 3, 0, from the line

$$\frac{x}{2}+\frac{y}{3}=1;$$

3. Find the distance of the point 0, 1, from the line

$$x-3y=1.$$

4. Find the distance of the point $-1$, 3, from the line

$$3x+4y+2=0.$$

5. Find the distance of the point $-a$, $-b$, from the line

$$\frac{x}{a}+\frac{y}{b}=1.$$

6. Find the distance of the point $a$, $b$, from the line

$$ax-by=0.$$

7. Find the distance of the point $h$, $k$, from the line

$$Ax+By+C=D.$$

8. Find the distance of the origin from the line

$$hx+ky=c^2.$$

9. Find the distance of the point $h$, $k$, from the line

$$hx+ky=c^2.$$

10. Find the distance of the point $a$, 0, from the line

$$y=mx+\frac{a}{m}.$$

11. Find the distance of the point $h$, $k$, from the line

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

12. Find the coordinates of the foot of the perpendicular in each of the preceding cases.

13. Find the distances between the lines

$$y = (x - a)\tan\alpha,$$

$$\text{and } y = (x - b)\tan\alpha;$$

$$\frac{x}{a} + \frac{y}{b} = 2, \quad \text{and } \frac{x}{a} + \frac{y}{b} = \frac{1}{2}.$$

14. Find the coordinates of the points of intersection of the four lines in questions 1—4.

15. Find the coordinates of the intersection of

$$x\cos\alpha + y\sin\alpha = p, \quad \text{and } x\cos\beta + y\sin\beta = p.$$

16. Find the coordinates of the angular points of the parallelogram whose sides have for their equations

$$x = a, \; x = b, \; x\cos\alpha + y\sin\alpha = p_1, \; x\cos\alpha + y\sin\alpha = p_2.$$

17. Find the equations to the diagonals of the parallelogram in question 16, the coordinates of their point of intersection, and the area of the parallelogram.

18. Three of the angular points of a parallelogram are $a$, $0$; $h$, $k$; $0$, $b$, respectively; find the coordinates of the other angular point, and the equations to the diagonals.

19. Determine the angles of the parallelograms in the preceding questions.

20. Determine the angle between the straight lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1,$$

and find the relation between $a$ and $b$ when this angle is 150°.

21. Determine the angle between the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ and } x \cos a + y \sin a = p.$$

22. Determine the angle between the lines

$$4r \cos \theta + 3r \sin \theta = 1,$$
$$r \cos \theta = 3.$$

23. If the axes are inclined at an angle $\omega$, and $\theta$ be the angle between the lines

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0,$$
$$\tan \theta = \frac{(A_1B_2 - A_2B_1) \sin \omega}{A_1A_2 + B_1B_2 - (A_1B_2 + A_2B_1) \cos \omega}.$$

24. $ABC$ is a triangle, $O$ the centre of the inscribed circle, $O_1$, $O_2$, $O_3$ the centres of the escribed circles; prove that $O_1CO_2$, $O_2AO_3$, $O_3BO_1$, $AOO_1$ are straight lines, and find their equations.

Take $A$ as origin, $AB$, $AC$ as axes.

25. Find the equation to the straight line which passes through the point $h, k$, and the intersection of

$$\frac{x}{a_1} + \frac{y}{b_1} = 1, \quad \frac{x}{a_2} + \frac{y}{b_2} = 1.$$

26. Find the equation to the straight line which passes through the intersection of

$$A_1x + B_1y + C_1 = 0, \; A_2x + B_2y + C_2 = 0,$$

and also through the origin.

27. Find the equation to the straight line which passes through the intersection of

$$\frac{x}{a} + \frac{y}{b} = 1, \quad y = mx,$$

and is perpendicular to the former line.

28. Find the equation to the straight line which passes through the intersection of

$$A_1x+B_1y+C_1=0,\quad A_2x+B_2y+C_2=0,$$

and is parallel to $A_3x+B_3y=0$.

29. Find the equation to the straight line which passes through the intersection of

$$x\cos\alpha+y\sin\alpha=p,\quad x\cos\beta+y\sin\beta=p,$$

and also of

$$y=mx+c,\ \text{and}\ x=my+c.$$

30. The perpendiculars from the angles of a triangle on the sides intersect in a point.

31. The straight lines intersecting the angles of a triangle intersect in a point.

32. If $ABC$ be a triangle, and $D$, $E$, $F$ points in the sides such that $BD : DC :: CE : EA :: AF : FB$, then $AD$, $BE$, $CF$ intersect in a point.

33. The locus of a point, the algebraic sum of whose distances from the sides of a polygon is constant, is a straight line.

# CHAPTER V.

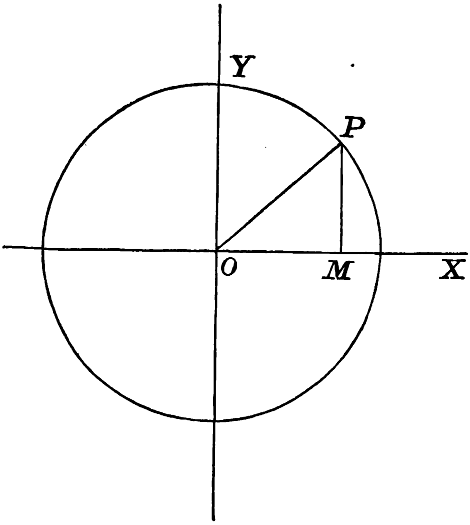
## THE CIRCLE.

**41.** Next in simplicity to the straight line comes the circle: we proceed to find its equation.

Let the axes be rectangular, and let the radius be $c$.

First, let the origin $O$ be the centre.

Let $P$ be any point on the circle, $PM$, $OM$ its coordinates.



Then $$OM^2 + PM^2 = OP^2,$$

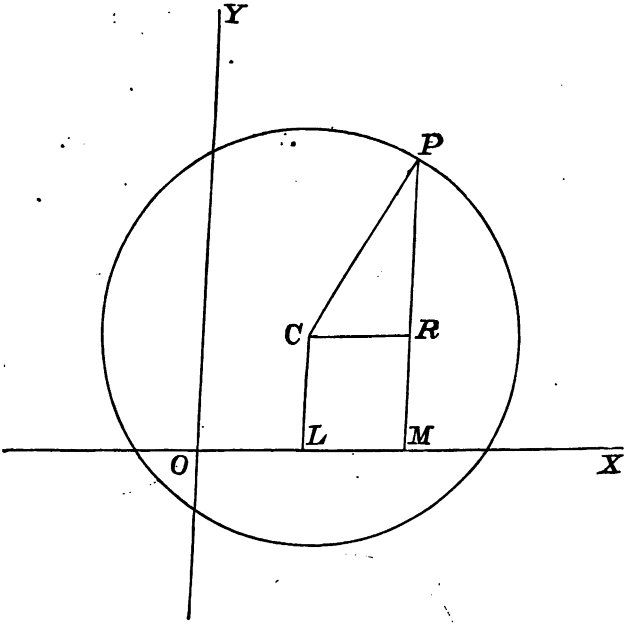
or, $$x^2 + y^2 = c^2,$$

the equation required.

**42.** Next, let the coordinates of the centre be $a$, $b$.

Then, in the figure,

$$CR^2 + PR^2 = CP^2;$$



but $$CR = LM = x - a,$$

$$PR = PM - CL = y - b;$$

$$\therefore (x-a)^2 + (y-b)^2 = c^2,$$

or $$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0,$$

the general equation to the circle referred to rectangular coordinates.

**43.** In this equation,

If $a=0$, $C$ lies on the axis of $y$, which is therefore a diameter.

If $b=0$, $C$ lies on the axis of $x$, which is therefore a diameter.

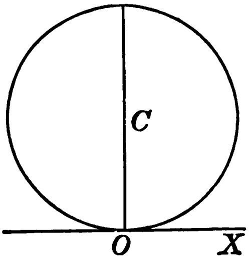
If $a=0$, $b=0$, we get the original equation

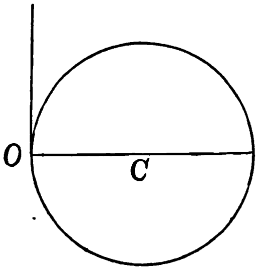
$$x^2+y^2=c^2.$$

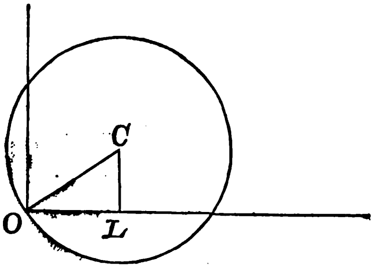
If $$a^2+b^2=c^2,$$

$$CP^2=OL^2+CL^2, \quad \therefore O \text{ lies on the curve.}$$

These cases are represented in the annexed figures.







**44.** *To find the condition that the general equation of the second degree*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \text{............}(1)$$

*may represent a circle.*

If possible, let it be the equation to that circle whose centre is the point $a$, $b$, and radius $c$.

Then the equation must be identical with

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - c^2 = 0.$$

Divide by $A$ and equate coefficients of $xy$, $y^2$ to the constant terms. Therefore

$$B = 0, \quad \frac{C}{A} = 1,$$

$$\frac{2D}{A} = -2a,$$

$$\frac{2E}{A} = -2b,$$

$$\frac{F}{A} = a^2 + b^2 - c^2.$$

The equation (1) will therefore represent a circle if $B = 0$, $C = A$, or the general equation to a circle is

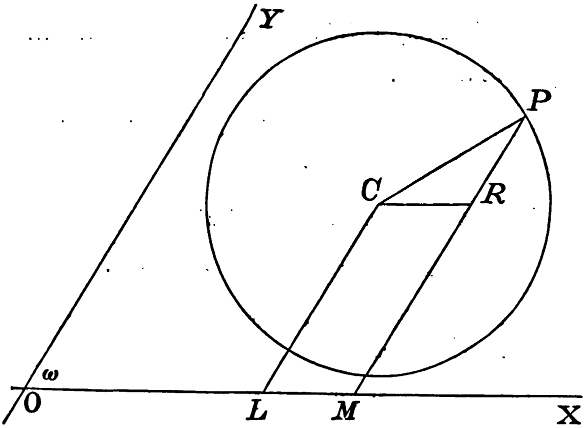
$$Ax^2 + Ay^2 + 2Dx + 2Ey + F = 0.$$

Hence, if the equation

$$x^2 + y^2 + 2Ax + 2By + C = 0$$

represent a circle, the coordinates of the centre are $-A$, $-B$ and the radius $(A^2 + B^2 - C)^{\frac{1}{2}}$.

**45.** *To find the equation to a circle referred to oblique axes.*



Let $a$, $b$ be the coordinates of the centre, $c$ the radius.

Here $CR^2 + RP^2 - 2CR \, . \, RP \cos CRP = CP^2$;

but $$CRP = \pi - YOX = \pi - \omega,$$

$$(x-a)^2 + 2(x-a)(y-b)\cos\omega + (y-b)^2 = c^2.$$

Expanding and rearranging, we obtain

$$x^2 + 2xy\cos\omega + y^2 - 2(a + b\cos\omega)x - 2(b + a\cos\omega)y$$
$$+ a^2 + 2ab\cos\omega + b^2 - c^2 = 0.$$

**46.** Here, as before, we find that the conditions that the general equation of the second degree

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

should represent a circle are

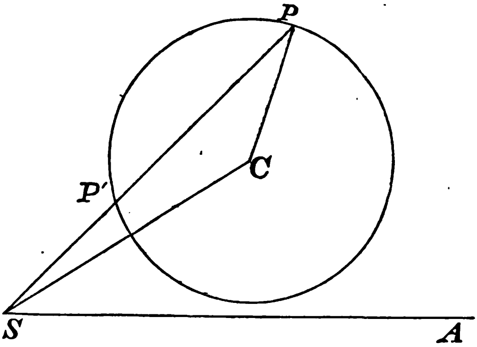
$$A = C, \ \frac{B}{A} = \cos\omega.$$

We shall, however, rarely use this equation.

**47.** *To find the equation to a circle referred to polar coordinates.*

First, let the centre be the pole, let the radius be $c$, then evidently $r = c$ is the equation required.

**48.** Next, let the coordinates of the centre $C$ be $l$, $\alpha$.



Then, in the triangle $SCP$,

$$SP^2 - 2SP \,.\, SC \cos PSC + SC^2 - CP^2 = 0,$$

or
$$r^2 - 2lr \cos(\theta - \alpha) + l^2 - c^2 = 0,$$

the equation required.

In this equation if $r_1$, $r_2$ be the two values $SP$, $SP'$ of $r$ corresponding to any value of $\theta$, we know that

$$r_1 r_2 = l^2 - c^2.$$

This proves that, if from any point a straight line be drawn cutting a circle, the rectangle contained by its segment is constant.

In this equation put $l = c$, then $S$ lies on the circle.

Then the absolute term vanishes, and the equation becomes

$$r - 2c \cos(\theta - \alpha) = 0.$$

## EXAMPLE V.

1. Determine the radii of the circles denoted by the following equations:

$$x^2 + y^2 = 9a^2,$$

$$(x + y)^2 + (x - y)^2 = 8a^2,$$

$$\frac{x^2 + y^2}{a^2 + b^2} = 1.$$

2. Determine the coordinates of the centre, and the radii of the circles denoted by the following equations:

$$x^2 + y^2 - 2a(x - y) = c^2,$$

$$x^2 + y^2 + ax + by = a^2 + b^2,$$

$$x^2 + y^2 - 3x - 4y + 4 = 0,$$

$$x^2 + y^2 = ax,$$

$$x^2 + y^2 = by,$$

$$x^2 + y^2 = a(x + a),$$

$$x^2 + y^2 = ax + by.$$

3. Find the equation to the circle whose radius is $a$, and coordinates of the centre $a$, $-a$.

4. To that whose radius is $5a$, and coordinates of centre $3a$, $4a$.

5. To that whose radius is $c$, and coordinates of centre $b + c$, $b - c$.

6. To that which passes through the origin and cuts off lengths $a$, $b$ from the axes.

7. To that which passes through the origin, and two given points $x_1y_1$, $x_2y_2$.

8. To that which passes through the three points $x_1y_1$, $x_2y_2$, $x_3y_3$.

9. To that which passes through the three points $a, 0$; $0, b$; $b, a$.

10. In the last four questions find the coordinates of the centre and the radius of each circle.

11. Find the equation to the circle which has its centre in the line $lx = my$, and cuts off chords of length $2a$, $2b$ from the axes.

12. Find the equation to the four circles whose radius is $\sqrt{2}a$, and which cut off chords from each axis equal to $2a$.

13. Find the equation to the circle which passes through the origin and cuts off equal lengths $a$ from the lines $y = x$, $x + y = 0$.

14. The equation to the circle whose centre is the origin and radius $a$, the axes being inclined at an angle $\omega$, is

$$x^2 + 2xy \cos \omega + y^2 = a^2.$$

15. Find the angle between the axes when the equation

$$x^2 + xy + y^2 - a(x + y) - b^2 = 0$$

represents a circle.

Find also its radius and the coordinates of its centre.

16. The axes being inclined at an angle $\omega$, find the equation to the circle which passes through the origin and the points $h, 0$; $0, k$.

17. Find the angle between the axes when the equation

$$x^2 + y^2 = a^2 + \sqrt{3}xy + ax$$

represents a circle: find the radius and coordinates of the centre.

18. If the equation

$$a^2(x + y - a)^2 = 2b^2xy$$

represent a circle, determine the angle between the axes, the centre, and the radius.

19. Prove analytically, Euc. III. Props. 3, 9.

20. Find the polar equation to the circle which has its centre at the pole and cuts off a chord of given length from the line

$$r \cos(\theta - \alpha) = p.$$

21. Prove, by using polar coordinates, Euc. III. 7, 8, 14, 15, 20, 21, 22, 31, 35.

22. A ladder is placed against a wall, shew that its middle point always lies on a circle.

23. If $A$, $B$ be fixed points, and $P$ a point such that $AP : BP$ in a constant ratio, $P$ lies on a circle, unless the ratio is equal to unity.

24. If two chords of a circle intersect at right angles, the sum of the squares of the segments is together equal to the square of the diameter.

25. Find the length of the chord of the circle

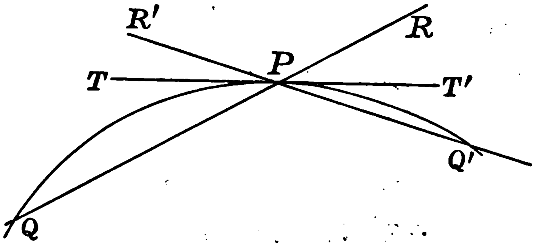
$$x^2 + y^2 = c^2,$$

the equation to the chord being $\frac{x}{a} + \frac{y}{b} = 1$.

## TANGENT AND NORMAL.

**49.** We might proceed to find the equation to the tangent to a circle at any point from the consideration that it is perpendicular to the diameter through that point; we will, however, give a definition of a tangent, and a method of finding its equation, which will be applicable to all curves.

**50.** DEF. Let $QPQ'$ be a curve, $P$ a point on it, $Q$ any other point on it; draw the secant $QPR$; let $Q$ move along the curve to $P$; then the limiting position of the secant $QPR$, when $Q$ *moves up to and ultimately coincides* with $P$, is called the tangent to the curve $QPQ'$ at $P$.



It is evident that on whichever side of $P$ we take $Q$, for every position of $Q$ there is a definite position of $PQ$; there must therefore be some position when the point $Q$ is neither on *one* side of $P$ nor on the *other*, but *at* $P$: this position is called the tangent at $P$.

**51.** DEF. The *normal* to a curve at any point is the perpendicular to the tangent through that point.

**52.** To find the equation to the tangent at any point of a circle.

Let the point be $x_1y_1$, and the equation to the circle

$$x^2+y^2=c^2 \quad\text{.........................}(1),$$

and first let us find the secant passing through the points $x_1y_1$, $x_2y_2$.

The equation to the straight line through these points has been already found, it is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \ldots\ldots\ldots\ldots\ldots (2).$$

Now if the points $P$, $Q$ coincide, $x_2 = x_1$; $y_2 = y_1$, and the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ assumes the form $\frac{0}{0}$.

We have not, however, introduced the condition that these points should lie on the circle.

Since $$x_1^2 + y_1^2 = c^2 = x_2^2 + y_2^2,$$

$$\therefore\ y_2^2 - y_1^2 = x_1^2 - x_2^2,$$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

Substituting in equation (2) we obtain

$$y - y_1 + \frac{x_2 + x_1}{y_2 + y_1}(x - x_1) = 0.$$

Now let $Q$ coincide with $P$, or $x_2 = x_1$, $y_2 = y_1$, and the equation becomes

$$y - y_1 + \frac{x_1}{y_1}(x - x_1) = 0,$$

or $$yy_1 + xx_1 = y_1^2 + x_1^2 = c^2:$$

$$xx_1 + yy_1 = c^2$$

is therefore the equation required.

If the centre be not the origin but the point $h$, $k$, transfer the origin to that point and we obtain

$$(x-h)(x_1-h)+(y-k)(y_1-k)=a^2.$$

**53.** *To find the points where the tangent cuts the axes.*

Putting $y$ and $x$ successively equal to zero in the equation to the tangent, we obtain

$$x=\frac{c^2}{x_1},\quad y=\frac{c^2}{y_1}.$$

**54.** *To find the equation to the normal at the point* $x_1y_1$.

Since the normal is perpendicular to the tangent

$$xx_1+yy_1=c^2,$$

its equation must be

$$(x-x_1)y_1-(y-y_1)x_1=0;$$

or

$$xy_1-yx_1=0.$$

The normal therefore passes through the centre.

**55.** *To find the condition that the straight line*

$$\frac{x}{a}+\frac{y}{b}=1$$

*should touch the circle*

$$x^2+y^2=c^2.$$

Eliminating $y$ by substituting $\frac{b}{a}(a-x)$ in the first of these equations, we obtain

$$x^2+b^2\left(\frac{a-x}{a}\right)^2=c^2;$$

or

$$(a^2+b^2)x^2-2ab^2x+a^2(b^2-c^2)=0.$$

If the straight line touches the circle, the values of $x$ obtained from this equation must be equal, since the tangent only meets the circle in one point.

But we know, that if the equation,

$$Ax^2 + Bx + C = 0,$$

has equal roots,

$$B^2 = 4AC;$$

$$\therefore\ a^2b^4 = a^2(a^2 + b^2)(b^2 - c^2),$$

or

$$a^2b^2 = (a^2 + b^2)c^2,$$

is the condition required.

**56.** *Otherwise*, if the straight line

$$\frac{x}{a} + \frac{y}{b} = 1$$

touches the circle, it must coincide with some tangent

$$xx_1 + yy_1 = c^2,$$

or

$$\frac{xx_1}{c^2} + \frac{yy_1}{c^2} = 1;$$

$$\therefore\ a = \frac{c^2}{x_1},\quad b = \frac{c^2}{y_1};\ \therefore\ x_1 = \frac{c^2}{a},\quad y_1 = \frac{c^2}{b};$$

$$\therefore\ c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1;$$

or

$$c^2(a^2 + b^2) = a^2b^2,$$

or

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}.$$

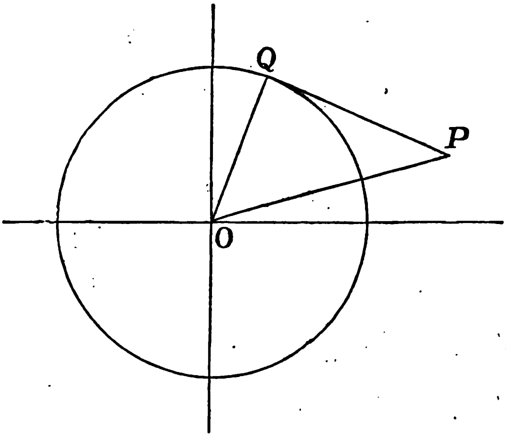
The straight line

$$x \cos \alpha + y \sin \alpha = c$$

evidently touches the circle, since it is at a distance from the centre equal to the radius.

**57.** *To find the length of the tangent drawn from the point* xy *to the circle*

$$x^2 + y^2 = c^2.$$



Let $PQ$ be the tangent, then

$$PQ^2 = OP^2 - OQ^2 = x^2 + y^2 - c^2.$$

Hence that expression which is equal to zero when the point $(xy)$, is on the circle, is equal to the square of the length of the tangent from $(xy)$ when the point is without the circle.

If the point be within the circle,

$$x^2 + y^2 - c^2 \text{ is negative,}$$

its square root is therefore impossible; this shews that no geometrical tangent can be drawn to the circle from a point within it.

## EXAMPLE VI.

1. Prove analytically, Euc. III. 16, 18, 19, 32, 36.

2. Write down the equations to the tangents to the circle

$$x^2 + y^2 = c^2$$

which pass through the points on the circle,

$$c,\ 0;\ \frac{3}{5}c,\ -\frac{4}{5}c;\ -h,\ k.$$

Find the equations to the tangents to the circle

$$x^2 + y^2 = c^2$$

which have the following properties respectively;

3. Make a given angle with the axis of $x$:

4. Are parallel to $\frac{x}{a} + \frac{y}{b} = 1$:

5. Are perpendicular to

$$Ax + By + C = 0:$$

6. Pass through a given point on the axis of $y$:

7. Are at a distance $\delta$ from the point $a \cos \alpha$, $a \sin \alpha$:

8. Cut off a given triangle from the axes.

9. Find the conditions that the lines

$$Ax + By + C = 0,$$
$$y = (x - x_1) \tan \alpha,$$
$$lx + my = d,$$

should touch the circle.

10. Shew that the equations to the tangents through the point $(hk)$ are

$$a\{hx+ky-(h^2+k^2)\}=\pm(kx+hy)(h^2+k^2-a^2)^{\frac{1}{2}}.$$

11. Shew that the equation to the tangent to the circle

$$x^2+y^2=ax+by,$$

which passes through the origin, is

$$ax+by=0.$$

12. Prove that the circles and lines whose equations are here given touch each other respectively, and find the points of contact in each case,

$$x^2+y^2+ax+by=0, \text{ and } ax+by+a^2+b^2=0;$$

$$x^2+y^2-2ax-2by+b^2=0, \text{ and } x=2a;$$

$$x^2+y^2-2c(x+y)+c^2=0, \text{ the axes, and}$$
$$(x-c)\cos\theta+(y-c)\sin\theta=c;$$

$$x^2+y^2=ax+by, \text{ and } ax-by+b^2=0.$$

13. Prove that the condition that the straight line

$$\frac{x}{a}+\frac{y}{b}=1$$

should touch the circle

$$x^2+Ax+y^2+By+C=0$$

is
$$4\left\{\frac{A}{a}+\frac{B}{b}+1+C\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}=\left(\frac{A}{b}+\frac{B}{a}\right)^2.$$

14. Find the condition that the straight line

$$x\cos\theta+y\sin\theta=p$$

should touch the circle

$$x^2+y^2=2(ax+by).$$

15. Prove that the tangent at the point $(x_1y_1)$ to the circle

$$x^2 + 2xy\cos\omega + y^2 = c^2$$

is

$$(x_1 + y_1\cos\omega)\,x + (y_1 + x_1\cos\omega)\,y = c^2.$$

16. Prove that the straight line

$$x\cos\alpha + y\cos\beta = c$$

will touch the circle

$$x^2 + 2xy\cos(\alpha+\beta) + y^2 = c^2.$$

17. Prove that the straight line

$$r\cos(\theta - \alpha) = a$$

touches the circle $r = a$ at the point $a$, $\alpha$.

18. Prove that the tangent to the circle

$$r = l\cos(\theta - \alpha),$$

at the point for which $\theta = \beta$, is

$$r\cos(\theta + \alpha - 2\beta) = l\cos^2(\beta - \alpha).$$

19. Find the tangents to the circle $r = l\cos(\theta - \alpha)$ at the pole, and at the extremity of the diameter through the pole.

20. Determine the point of contact when the tangent makes an angle $\gamma$ with the initial line.

## POLES AND POLARS.

**58.** *From a given point tangents are drawn to a circle, to find the equation to the straight line passing through the points of contact.*

Let $h$, $k$ be the coordinates of the given point, let $x_1$, $y_1$ be the coordinates of one point of contact, then the equation to the tangent at $(x_1y_1)$ is

$$xx_1 + yy_1 = c^2;$$

but, since this passes through $(h, k)$

$$hx_1 + ky_1 = c^2.$$

Similarly, if $x_2$, $y_2$ be the coordinates of the other point of contact,

$$hx_2 + ky_2 = c^2,$$

since $(h, k)$ lies on the tangent through $x_2 y_2$.

Hence, since

$$hx_1 + ky_1 = c^2,$$

$$hx_2 + ky_2 = c^2,$$

$x_1$, $y_1$, $x_2$, $y_2$ satisfy the equation

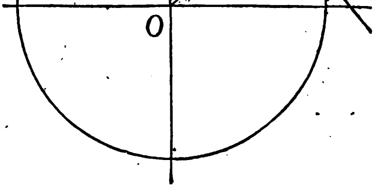
$$hx + ky = c^2;$$

this, however, is the equation to a straight line, and since it is satisfied by the coordinates of two points, $(x_1 y_1)$, $(x_2 y_2)$, it is the equation of the straight line which passes through those points: it is therefore the equation to the straight line required.

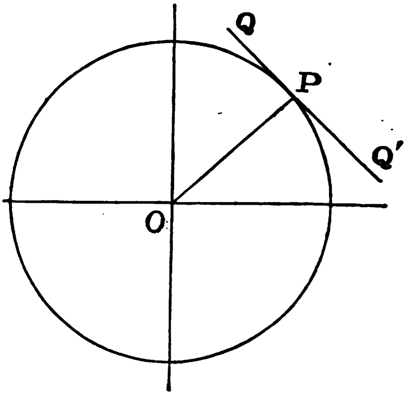
**59.** We have seen that when the point $(h, k)$ is without the circle, the equation

$$hx + ky = c^2$$

represents the chord of contact of tangents through $(h, k)$; when $(h, k)$ is on the circle, the same equation represents the tangent at $(h, k)$; what will then this equation represent when the point $(h, k)$ is within the circle, so that no real tangents can be drawn through it to the circle?



It is still the equation to a straight line, and since its form is unchanged whatever be the *position* of the point $(h, k)$ the equation must represent some geometrical facts which are independent of that *position*.



Let $P$ be the point $(h, k)$ either without, on, or within the circle, then the equation to $OP$ is

$$\frac{x}{h} = \frac{y}{k} \ldots\ldots(1);$$

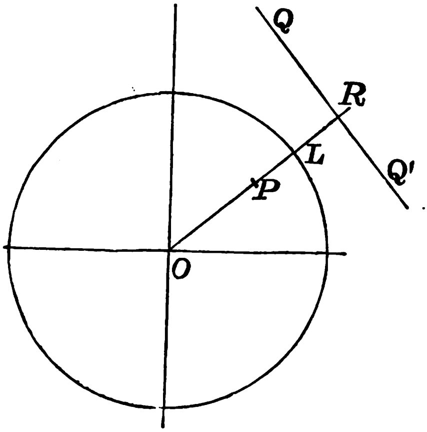
$\therefore$ (Art. 27), the line

$$hx + ky = c^2 \ldots\ldots(2)$$

is perpendicular to $OP$.

Again, the distance of the origin from the line (2) is

$$\frac{c^2}{(h^2 + k^2)^{\frac{1}{2}}}, \text{ (Art. 33).}$$



Now,

$$OP = (h^2 + k^2)^{\frac{1}{2}}, \; OL = c;$$

therefore if in $OP$ or $OP$ produced we take a point $R$ such that

$$OR : OL :: OL : OP,$$

and through $R$ draw $QRQ'$ perpendicular to $OP$, the equation to $QRQ'$ will be

$$hx + ky = c^2.$$

**60.** $QRQ'$ is called the *polar* of $P$: conversely, $P$ is called the *pole* of $QRQ'$.

The polar of a point may be defined either geometrically or algebraically.

*Geometrically*, thus: let $O$ be the centre of a circle, $P$ any point, join $OP$ and divide it, produced if necessary, in $Q$, so that $OR$ is a third proportional to $OI$ and the radius: through $R$ draw a straight line $QRQ'$ at right angles to $CP$: this straight line is called the polar of $P$.

Conversely, let $QQ'$ be any straight line, draw $OR$ perpendicular to $QQ'$, and in $OR$, produced if necessary, take a point such that $OP$ is a third proportional to $OR$ and the radius, then $P$ is called the pole of $QQ'$.

*Algebraically:* let the coordinates of any point be $h$, $k$; then the straight line represented by the equation

$$hx + ky = c^2$$

is called the polar of $(h, k)$.

Conversely, let the equation to any straight line be thrown into the form

$$hx + ky = c^2,$$

then $(h, k)$ is the pole of the line.

Thus; required the pole of

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Multiply by $c^2$,

$$\therefore \frac{c^2}{a}x + \frac{c^2}{b}y = c^2,$$

$\therefore \dfrac{c^2}{a}, \dfrac{c^2}{b}$, are the coordinates of the pole.

The student is recommended to pay particular attention to the preceding articles; there is no part of the subject which it is more necessary to understand thoroughly.

**61.** *If the polar of* Q *passes through* P, *then* Q *lies on the pole of* P.

Let the coordinates of $P$ be $h$, $k$, of $Q$, $x'$, $y'$, then, since $Q$ is on the polar of $P$, $Q$ lies on the line

$$hx + ky = c^2 \;\dots\dots\dots\dots\dots\dots \;(1),$$

$$\therefore\; hx' + ky' = c^2 \;\dots\dots\dots\dots\dots\dots \;(2).$$

But the polar of $(x', y')$ is

$$xx' + yy' = c^2 \;\dots\dots\dots\dots\dots\dots \;(3).$$

In this equation if we put $x = h$, we get from (2), $y = k$.

Therefore the point $(h, k)$ lies on the line (3), and conversely the line (3) passes through the point $(h, k)$.

**62.** Similarly, if the polar of $Q$ passes through $P$, then $Q$ lies on the pole of $P$.

Let the coordinates of $P$ be $h$, $k$, of $Q$, $x'$, $y'$, then the polar of $Q$ is

$$xx' + yy' = c^2;$$

but this equation represents a straight line which passes through $(h, k)$,

$$\therefore\; hx' + ky' = c^2;$$

$\therefore$ $x'$, $y'$, are the coordinates of a point which satisfies the condition

$$hx + ky = c^2,$$

that is, the point $Q$ lies on the polar of $P$.

**63.** If the origin be not the centre, and the point $(a, b)$ be the centre, we must in these equations write $x-a$, $y-b$, for $x$ and $y$.

**64.** To find the polar of the origin with respect to the circle

$$(x-a)^2+(y-b)^2=c^2.$$

The polar of the point $(h, k)$ is

$$(x-a)(h-a)+(y-b)(k-b)=c^2,$$

or $$(h-a)x+(k-b)y-ha-kb+a^2+b^2-c^2=0.$$

Put $h=0$, $k=0$, then

$$ax+by=a^2+b^2-c^2,$$

the equation required.

## Two or more Circles.

**65.** *To find the equation to the straight line which passes through the intersection of two circles which cut each other.*

Let the equations to the circles be

$$x^2+y^2-2a_1x-2b_1y+a_1^2+b_1^2-c_1^2=0 \text{.........} (1),$$

$$x^2+y^2-2a_2x-2b_2y+a_2^2+b_2^2-c_2^2=0 \text{.........} (2).$$

Subtract one of these equations from the other: then

$$2(a_1-a_2)x+2(b_1-b_2)y=a_1^2-a_2^2+b_1^2-b_2^2-(c_1^2-c_2^2) \text{......} (3).$$

This is the equation to the straight line required, for it is the equation to *some* straight line, and it passes through the intersection of (1) and (2), because whenever equations (1) and (3) are satisfied simultaneously, equation (2) is satisfied also.

**66.** *Let the expressions on the left-hand side of equations* (1), (2) *be denoted by* $S_1$, $S_2$, *respectively, then equation* (3) *may be written*

$$S_1 - S_2 = 0, \text{ or } S_1 = S_2 \text{ ..................... } (4).$$

Let us investigate the geometrical meaning of this expression.

We have seen that when the circles cut one another the chord of intersection is represented by equations (3) or (4); they may however not cut one another, in which case (3) or (4) will still represent a straight line.

Now we know (Art. 57) that $x^2 + y^2 - c^2$ is the square on the tangent drawn from $(x, y)$ to the circle

$$x^2 + y^2 - c^2 = 0.$$

Similarly, $(x - a)^2 + (y - b)^2 - c^2$ is the square on the tangent drawn from $(x, y)$ to the circle

$$(x - a)^2 + (y - b)^2 - c^2 = 0,$$

or $S$ is the square of the tangent drawn from $(x, y)$ to the circle $S = 0$.

Hence the meaning of equation (4) is, that for all points represented by it, the squares on the tangents, and therefore the tangents themselves drawn from them to the circles, are equal to one another.

Hence, if we subtract the expression on the left-hand side of the equation to one circle from the similar expression in the equation to another circle we get the equation to a straight line, and, if from any point in this straight line we draw tangents to the circles, these tangents will be equal to one another.

This straight line is called the *radical axis* of the two circles.

**67.** Let the equations to three circles be

$$S_1 = (x - a_1)^2 + (y - b_1)^2 - c_1^2 = 0,$$
$$S_2 = (x - a_2)^2 + (y - b_2)^2 - c_2^2 = 0,$$
$$S_3 = (x - a_3)^2 + (y - b_3)^2 - c_3^2 = 0.$$

Then the equations to the radical axes of these circles taken two and two together are

$$S_1 - S_2 = 0,$$
$$S_2 - S_3 = 0,$$
$$S_3 - S_1 = 0.$$

At the point where the straight lines represented by the first two of these equations intersect, we have

$$S_1 - S_2 = S_3 - S_2 = 0,$$

or

$$S_3 - S_1 = 0,$$

that is, the third straight line passes through the intersection of the first two.

Hence the three radical axes of any three circles meet in a point.

This point is called the *radical centre* of the three circles.

**68.** In the equation to the circle

$$(x - a)^2 + (y - b)^2 - c^2 = 0.$$

Let $c = 0, \quad \therefore x - a = 0, \; y - b = 0,$

the equation therefore represents the point $(a, b)$.

A point may therefore be considered as a circle of infinitely small radius. If in the equations to two circles we put the radius of one equal to zero, and subtract, we obtain the equation to the radical axis as before, which is now a straight line such that tangents from any point in it to a circle are equal in length to the distance of that point from a given point.

If both radii vanish, and both circles become points, the radical axis becomes the straight line, every point in which is equally distant from the two given points.

**69.** *To find the common tangents to two circles.*

Let the equations to the two circles be

$$x^2 + y^2 = c_1^2 \dots\dots\dots\dots (1),$$

$$(x - a)^2 + y^2 = c_2^2 \dots\dots\dots\dots (2).$$

Let $$x \cos\theta + y \sin\theta = p$$

be the equation to the common tangent.

Then since this straight line touches (1),

$$p = \pm c.$$

Also, since it touches (2), the distance from the point $a$, $0$, must be $c_2$.

But this distance is

$$\pm (a \cos\theta - p),$$

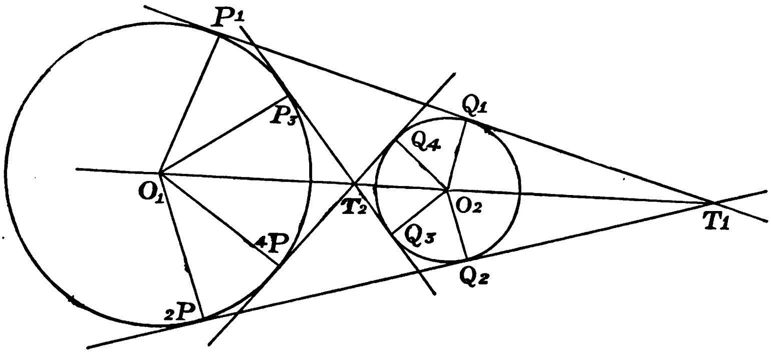
$$\therefore\ \pm (a \cos\theta \mp c_1) = c_2;$$

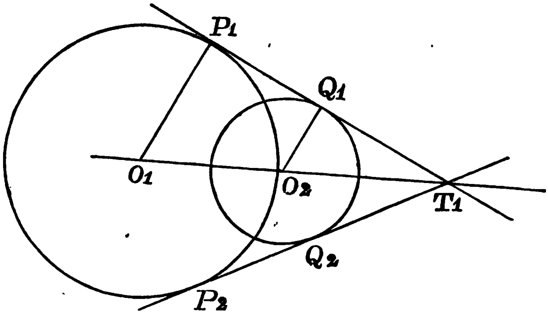
$$\therefore\ \cos\theta = \pm \frac{c_1 \pm c_2}{a};$$

$$\sin\theta = \frac{\{a^2 - (c_1 \pm c_2)^2\}^{\frac{1}{2}}}{a}.$$

Hence the equations to the tangents may be written

$$x (c_1 \pm c_2) \pm y \{a^2 - (c_1 \pm c_2)\}^{\frac{1}{2}} = ac_1.$$





**70.** In the above equations, first, let $a > c_1 + c_2$, or

$$O_1O_2 > O_1P_1 + O_2Q_2,$$

therefore the circles do not meet, hence there can be *four* common tangents drawn to them.

$$x(c_1 + c_2) \pm y\,\{a^2 - (c_1 + c_2)^2\}^{\frac{1}{2}} = ac,$$

are the equations to the internal tangents which cut the axis of $x$ in the point $T_2$ such that

$$O_1T_2 = \frac{O_1P_2}{O_1P_3 + O_2Q_3}\,O_1O_2,$$

$$\text{or } x = \frac{c_1}{c_1 + c_2} a\,;$$

and $$x(c_1 - c_2) \pm y\{a^2 - (c_1 - c_2)^2\}^{\frac{1}{2}} = ac_1,$$

the equations to the external tangents which cut $O_1O_2$ produced at a point $T_1$ such that

$$OT_1 = \frac{c_1}{c_1 - c_2} a.$$

$O_1$, $O_2$, are called the external and internal centres of similitude.

If $c_1 + c_2 > a$, the two circles cut one another, and only two tangents can be drawn to touch both circles.

This is shewn analytically by one of the values of $\cos\theta$ becoming greater than 1, which shews that no angle exists having the required cosine.

If $c_1 - c_2 > a$, both values of $\cos\theta$ are greater than 1; in this case all the equations become irrational, or no such tangent can be drawn.

In this case, since

$$O_1O_2 < O_1P_1 - O_2Q_1,$$

$$O_1O_2 + O_2Q_1 < O_1P_1,$$

and one circle lies entirely within the other.

## EXAMPLE VII.

1. Write down the polars with respect to the circle

$$x^2 + y^2 = c^2$$

of the following points,

$$(c, \ c); \ (2c, \ 3c); \ (a + b, \ a - b).$$

2. Find the poles of the following lines:

$$Ax + By + C = 0; \quad \frac{x}{a} + \frac{y}{b} = 2; \quad y - y_1 = (x - x_1) \tan\theta;$$

$$y = b; \quad x = a.$$

3. If the pole always lie on the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the equation to the polar is

$$(ax - c^2) \cos\theta + (by - c^2) \sin\theta = 0,$$

where $\theta$ is any angle.

4. If the pole lie on the circle

$$x^2 + y^2 = 4c^2,$$

the polar will touch the circle

$$x^2 + y^2 = \frac{c^2}{4}.$$

5. Find the equation to the circle whose diameter is the common chord of the circle

$$x^2 + y^2 = c^2, \text{ and } (x - a)^2 + y^2 = c^2.$$

6. Find the equation to the straight line which joins the centres of

$$x^2+2x+y^2=0,\quad y^2+2y+x^2=0.$$

7. Find also the tangents at the origin to these two circles.

8. Find the length of the common chord of the two circles

$$(x-a)^2+(y-b)^2=c^2,$$
$$(x-b)^2+(y-a)^2=c^2,$$

and hence prove that the condition they should touch each other is

$$2c^2=(a-b)^2.$$

9. Prove that all circles represented by the equation

$$(x-a_1)^2+(y-b_1)^2+k\{(x-a_2)^2+(y-b_2)^2\}=c^2,$$

where $k$ is any constant, have the same radical axis.

10. Find the radical axes and radical centre of the circles

$$(x-1)^2+(y-2)^2=6,$$
$$(x-2)^2+(y-3)^2=8,$$
$$(x-3)^2+(y-1)^2=10.$$

11.* The abscissæ of the centres of two circles are $a_1$, $a_2$, and the lengths of tangents from the origin $l_1$, $l_2$, respectively; shew that the radical axis cuts the axis of $x$ at the point whose abscissa is

$$\frac{l_2^2-l_1^2}{2(a_1-a_2)}.$$

12. A fixed circle is cut by a series of circles, all of which pass through two points: shew that the radical centre is a fixed point.

13. Three circles have fixed centres, and their radii are $r_1+\rho$, $r_2+\rho$, $r_3+\rho$, where $\rho$ is variable; shew that the radical centre lies on a straight line.

14. Find the equations to the four common tangents to the circles

$$x^2+y^2=4c^2, \quad x^2+y^2-8cx+15c^2=0.$$

15. Prove that the circles

$$x^2+y^2=c^2; \quad x^2+y^2+a^2+2ac=2(a+c)y,$$

have three common tangents, and find their equations.

16. Prove that the circles

$$x^2+y^2=(c+a)^2, \quad (x-a)^2+y^2=c^2$$

have only one common tangent, and find its equation.

17. Find the locus of the intersection of a perpendicular from a fixed point on a line passing through another fixed point.

18. $A$ is a given point, $P$ a point on a fixed circle: $AP$ is divided in $Q$ so that $AQ : PQ$ in a given ratio; shew that the locus of $Q$ is a circle.

19. Find the locus of a point such that its distance from the line

$$x\cos\alpha+y\sin\alpha=p$$

is a third proportional to a given line $b$, and its distance from a given point.

20. Find the locus of the intersection of two straight lines, which pass, each through a given point, and are inclined at a given angle.

21. The four tangents, which are common to two circles which do not intersect, and are terminated at the points of contact, have their centres on the radical axis.

22. Given one side of a triangle and the opposite angle, find the locus of the intersection of the bisectors of the other angles.

23. $AB$ is a given diameter of a circle of which $C$ is the centre, $DE$ a chord such that $DCE = 2\alpha$; $AD$, $BE$ intersect in $P$: prove that the locus of $P$ is a circle.

24. From a fixed point $C$ a straight line $PCP'$ is drawn cutting a fixed circle in $P$, $P'$, find the locus of $Q$ when

$$CQ^2 = CP . CP'.$$

25. $AB$, $AC$ are fixed straight lines, $DE$ a line of fixed length terminated by them in $D$, $E$, from $D$, $E$ are drawn $DP$, $EP$ perpendicular to $AD$, $AE$, intersecting in $P$; find the locus of $P$.

26. A straight line through a point $R$ cuts two fixed circles in $PP'$, $QQ'$ respectively; if

$$RP . RP' = \mu RQ . RQ',$$

find the locus of $R$.

27. A circle of given radius moves in such a manner with respect to a fixed circle, that their radical axis always passes through a fixed point; shew that the locus of the centre of the moving circle is a circle, of which the given fixed point is centre.

28. Shew that the locus of the middle points of the chords of one circle which touch another circle, is

$$(x^2 + y^2 - ax - by)^2 = d^2 (x^2 + y^2),$$

the equation to the circles being

$$x^2 + y^2 = c^2,$$

$$(x - a)^2 + (y - b)^2 = d^2.$$

29. Shew that the equation to a chord of the circle

$$r = c\cos\theta,$$

is
$$\frac{c\cos\alpha\cos\beta}{r} = r\cos(\alpha+\beta-\theta),$$

where $\alpha$, $\beta$, are the angular coordinates of the extremities of the chord.

30. Two circles whose radii are $a$, $b$, cut at an angle $\alpha$, the length of the common chord is

$$\frac{2ab\sin\alpha}{(a^2+2ab\cos\alpha+b^2)^{\frac{1}{2}}}.$$

# CHAPTER VI.

## THE PARABOLA.

**71.** A conic section may be defined as the locus of a point, the distance of which from a given point is always in a constant ratio to its distance from a given straight line.

When this ratio is less than unity, the curve is called an *ellipse;* when equal to unity, a *parabola;* when greater than unity, a *hyperbola.*

The fixed point is called the *focus,* the fixed straight line the *directrix,* and the given ratio the *excentricity* of the conic.

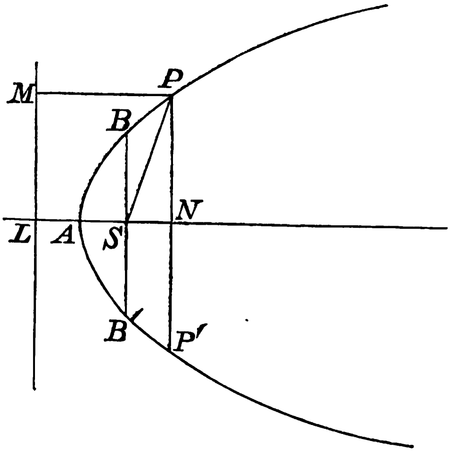
Conic sections are so called because when a right cone is cut by a plane, the section is always one of these curves.

**72.** *To find the polar equation to a conic section.*

Let $S$ be the focus, $LM$ the directrix, draw $SL$ perpendicular to $LM$, and take $S$ as the pole, $SL$ as the initial line; let $SL = c$, $P$ be any point on the conic, $e$ the excentricity.

Then $$SP = e \,.\, PM = e \,.\, L\overset{N}{\cancel{M}} = e \,.\, (LS + SN),$$

or $$r = e\,(c + r\cos\theta),$$



$$\therefore\ r\,(1 - e\cos\theta) = ec,$$

$$r = \frac{ec}{1 - e\cos\theta}.$$

Let $$\theta = \pm\frac{\pi}{2};\ \therefore\ r = ec,$$

or *BSB'* the chord through *S* at right angles to *LS* is equal to $2ec$.

This chord is called the latus rectum; if its length be denoted by $2l$, the equation to the curve may be put in the form

$$r = \frac{l}{1 - e\cos\theta},\ \text{or}\ \frac{l}{r} = 1 - e\cos\theta,$$

the equation required.

**73.** *To find the equation to the parabola referred to rectangular axes.*

In the preceding figure, take the directrix $LM$ as the axis of $y$, and the straight line perpendicular to it through the focus as axis of $x$, let

$$LS = 2a.$$

Then
$$SP = MP = LN,$$

$$\therefore SP^2 = LN^2,$$

$$\text{or, } SN^2 + PN^2 = LN^2,$$

$$\text{that is, } (x - 2a)^2 + y^2 = x^2,$$

$$\text{or, } y^2 = 4a(x - a),$$

the equation required.

If in this equation we put $y = 0$, $x = a$; this shews that the curve cuts the axis at a point $A$, such that $LS$ is bisected in $A$, this point is called the *vertex*, and $AS$ produced the *axis* of the curve.

If we transform the equation to parallel axes through $A$, the axis of $x$ is unchanged and that of $y$ is at a distance $a$ from the old axis of $y$, hence

$$y = y', \quad x = x' + a,$$

and the equation becomes

$$y'^2 = 4ax',$$

or, suppressing the accents,

$$y^2 = 4ax.$$

This is therefore the equation to the parabola referred to its vertex as origin, and axis as axis of $x$.

**74.** *To trace the form of the parabola from its equation.*

Since $y^2 = 4ax$, or

$$x = \frac{y^2}{4a},$$

$x$ cannot be negative, that is, the curve lies wholly on the positive side of the axis of $y$.

Since $y^2 = 4ax$, $y = \pm 2(ax)^{\frac{1}{2}}$,

therefore, since this equation is unaltered if we write $-y$ for $y$, to every point $P$ on the curve on the positive side of the axis there corresponds another point $P'$, on the negative side, such that $P'N = PN$.

The curve is therefore symmetrical with respect to the axis of $x$.

Again, if $x = 0$, $y = 0$, and has no other value, therefore the curve does not cut either axis at any other point besides the origin.

Also, the greater value we give to $x$, the greater values we get for $y$, and when $x$ is infinite $y$ is infinite, hence the curve goes off to an infinite distance on each side of the axis of $x$.

**75.** *To find the distance of any point from the focus.*

Let $P$ be any point on the curve,

then $$SP^2 = SN^2 + PN^2 = (x-a)^2 + y^2$$
$$= (x-a)^2 + 4ax = (x+a)^2,$$
$$\therefore SP = a + x.$$

**76.** *To find the length of the latus rectum.*

In the equation

$$y^2 = 4ax,$$

put $x = a$;

then $y^2 = 4a^2$, $y = \pm 2a$,

or the latus rectum $BB' = 4a$.

**77.** *To find the equation to the tangent at any point of a parabola.*

Since the equation to the parabola is essentially unsymmetrical, it is convenient in problems respecting it to use the unsymmetrical form of the equation to a straight line,

$$y = mx + c.$$

Let $(x_1, y_1)$, $(x_2, y_2)$ be two points on a parabola, $P$ and $Q$, then the equation to the straight line joining them is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1):$$

but since these points are on the parabola,

$$y_2^2 = 4ax_2, \quad y_1^2 = 4ax_1;$$

$$\therefore\ y_2^2 - y_1^2 = 4a(x_2 - x_1);$$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1};$$

therefore the equation to the straight line joining $(x_1, y_1)$, $(x_2, y_2)$, becomes

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1).$$

In this equation, multiply by $y_1 + y_2$,

then $$y(y_1 + y_2) - y_1^2 - y_1y_2 = 4a(x - x_1)$$

or, remembering that $y_1^2 = 4ax_1$

$$y(y_1 + y_2) - y_1y_2 = 4ax.$$

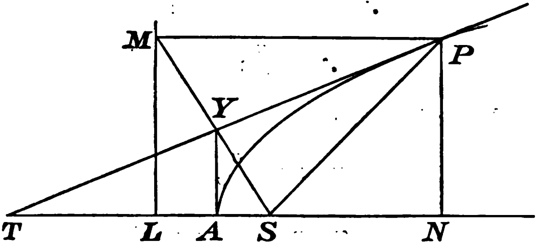
Now let $Q$ move up to $P$, and therefore $x_2 = x_1$, $y_2 = y_1$, then $PQ$ becomes the tangent at $P$, and the equation becomes

$$2yy_1 - y_1^2 = 4ax,$$

$$\text{or, } 2yy_1 = 4a(x + x_1),$$

$$\therefore yy_1 = 2a(x + x_1)$$

the equation required.



**78.** In this equation put $y = 0$,

$$\therefore x = -x_1,$$

therefore the tangent $PT$ cuts the axis at a point $T$ such that $AT = AN$.

Now $$ST = AS + AT = a + x_1 = SP;$$

$$\therefore STP = SPT.$$

Again, $$\tan PTA = \frac{PN}{TN} = \frac{y_1}{2x_1} = \frac{2ay_1}{4ax_1} = \frac{2a}{y_1}.$$

**79.** *To find the equation to the tangent to a parabola in terms of the tangent of the angle it makes with the axis.*

Let the line $y = mx + c$

*cut* the parabola $y^2 = 4ax$,

then we shall find the abscissæ of the points of intersection by substituting for $y$ in this equation, and finding the roots of the quadratic

$$(mx + c)^2 = 4ax.$$

Now if the straight line *touch* the parabola, these roots must be equal;

$$\therefore m^2x^2 + 2(mc - 2a)x + c^2 = 0,$$

must have equal roots;

$$\therefore m^2c^2 = (mc - 2a)^2;$$

$$\therefore c = \frac{a}{m};$$

therefore the straight line

$$y = mx + \frac{a}{m},$$

or $$y = x \tan\theta + a \cot\theta,$$

touches the parabola $y^2 = 4ax.$

**80.** *To find the coordinates of the point of contact.*

Let $(x \tan\theta + a \cot\theta)^2 = y^2 = 4ax,$

$$\therefore x \tan\theta - a \cot\theta = 0,$$

$$x = a \cot^2\theta.$$

Similarly $y = 2a \cot\theta.$

This form of the equation is often useful in problems which do not involve the coordinates of the point of contact.

Example:

Tangents at right angles intersect in the directrix.

Let $$y = mx + \frac{a}{m},$$

$$y = nx + \frac{a}{n},$$

be the equations to the tangents; then since they are at right angles to each other,

$$n = -\frac{1}{m},$$

$$\therefore y = -\left(\frac{x}{m} + ma\right),$$

is the equation to the tangent at right angles to the first.

Subtract this equation from the first, then

$$(x + a)\left(m + \frac{1}{m}\right) = 0,$$

since this is to be true for all values of $m$,

$$\therefore x = -a,$$

the equation to the directrix.

These straight lines therefore intersect on the directrix.

**81.** *To find the equation to the normal at any point of a parabola.*

Let $x_1$, $y_1$ be the coordinates of the point, then

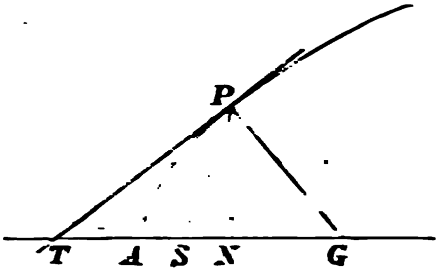
$$yy_1 = 2a(x + x_1),$$

is the equation to the tangent at $(x_1, y_1)$, therefore the equation to the straight line through $(x_1, y_1)$ perpendicular to this tangent must be

$$2a(y - y_1) + y_1(x - x_1) = 0,$$

the equation to the normal required.

**82.** Let $PT$, $PG$, be the tangent and normal at $P$, then $TN$ is called the subtangent, $NG$ the subnormal.



In the equation to the normal

$$2a(y - y_1) + y_1(x - x_1) = 0,$$

put $y = 0$;

$$\therefore x - x_1 = 2a, \quad \text{or } NG = 2a.$$

Now $SG = SN + NG = x - a + 2a = x + a = SP = ST$;

therefore the circle described on $GT$ as diameter will pass through $P$ and have $S$ for its centre.

83. *To find the equation to the normal in terms of the tangent it makes with the axis of x.*

The equation to the normal may be written

$$y = -\frac{y_1}{2a}x + y_1 + \frac{x_1 y_1}{2a}.$$

Let $-\frac{y_1}{2a} = m$, then

$$y_1 = -2am, \quad \frac{x_1 y_1}{2a} = \frac{y_1^3}{4a^2} = -2m^3a;$$

$$\therefore y = m\{x - 2a(1 + m^2)\},$$

the equation required.

**84.** In the equation to the tangent,

$$yy_1 = 2a(x + x_1).$$

Let $x_1$, $y_1$ each be zero, then the equation becomes $x = 0$, or the axis of $y$ touches the curve.

**85.** *To find the locus of the point in which the perpendicular from the focus on the tangent meets that tangent.*

Let the equation to the tangent be

$$y = mx + \frac{a}{m} \quad \dotfill \quad (1),$$

then the equation to the straight line through $(a, 0)$ perpendicular to (1) is

$$my + \frac{(x - a)}{\cancel{c}} = 0, \text{ or } y + \frac{x}{m} = \frac{a}{m};$$

at the intersection of these two straight lines we must have

$$x\left(m + \frac{1}{m}\right) = 0, \quad \therefore \ x = 0.$$

This is the equation to the tangent at the vertex: hence the locus of the foot of the perpendicular from the focus on a tangent is the tangent at the vertex.

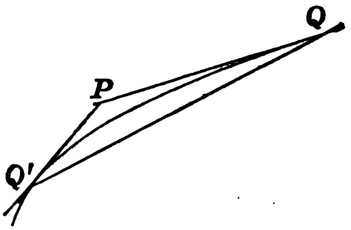
**86.** *To find the equation of the chord of contact of tangents drawn through a given point.*

Let the coordinates of $P$ be $h$, $k$, of $Q$ and $Q'$ $x_1$, $y_1$, $x_2$, $y_2$, respectively, then the equation to $PQ$ is

$$yy_1 = 2a(x + x_1),$$

and to $PQ'$,

$$yy_2 = 2a(x + x_2).$$



Now since $(h, k)$ lies on each of these lines, we must have

$$ky_1 = 2a(h + x_1),$$

$$ky_2 = 2a(h + x_2),$$

that is, $(x_1, y_1)$, $(x_2, y_2)$, satisfy the condition

$$kx = 2a(x + h).$$

Now this is the equation to some straight line, and the coordinates of $Q$ and $Q'$ satisfy it, it is therefore the equation to $QQ'$.

$P$ is called the pole of $QQ'$, $QQ'$ the polar of $P$, and whether the point $(h, k)$ be within, on, or without the parabola, the straight line represented by the equation

$$ky = 2a(x + h)$$

is called the polar of $(h, k)$.

**87.** *To find the pole of a given line.*

Let the equation to the straight line be of the form

$$y = mx + c.$$

Divide by $m$ and multiply by $2a$,

$$\therefore \frac{2a}{m}y = 2ax + \frac{2ac}{m};$$

comparing this with the equation to the polar, we get

$\dfrac{2a}{m}, \dfrac{c}{m}$ as the pole of the given line.

**88.** By reasoning precisely similar to that by which we proved the analogous proposition in the case of the circle, we may prove that if $P$ be the polar of $QQ'$, then the polar of every point on $QQ$ passes through $P$.

Example. The polar of the focus $a$, 0, is

$$0 = x + x_1, \text{ or } x = -x_1,$$

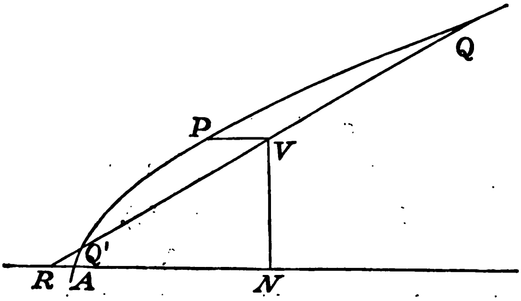
the directrix.

Hence the polar of every chord through the focus lies on the directrix: hence also the directrix is the locus of the intersection of tangents at the extremities of focal chords: but we have proved it to be the locus of tangents which intersect at right angles: hence tangents at the extremities of focal chords intersect at right angles in the directrix.

## DIAMETERS.

**89.** Let $QQ'R$ be a straight line cutting the parabola in $Q$, $Q'$.

Let $QRA = \theta$. Let $QV = Q'V$.



Let the equation to $QQ'$ be

$$y = (x - x_1) \tan\theta;$$

substituting for $x$ in the equation to the parabola we obtain

$$y^2 = 4a (y \cot\theta + x_1).$$

Now if $y_1$, $y_2$ be the roots of this equation,

$$\frac{y_1 + y_2}{2} = VN,$$

but

$$y_1 + y_2 = 2a \cot\theta,$$

$$\therefore VN = 2a \cot\theta,$$

therefore the straight line

$$y = 2a \cot\theta, \text{ or } y \tan\theta = 2a,$$

bisects all chords of the parabola which are parallel to the line

$$y = x \tan\theta.$$

Hence the locus of a system of parallel chords in a parabola is a straight line parallel to the axis. Every such straight line is called a diameter of the curve.

**90.** The coordinates of the point where the diameter

$$y = 2a \cot\theta$$

meets the curve

$$y^2 = 4ax,$$

are evidently

$$a \cot^2\theta, \ 2a \cot\theta,$$

but the equation to the tangent at this point is

$$2ay \cot\theta = 2a (x + a \cot^2\theta),$$

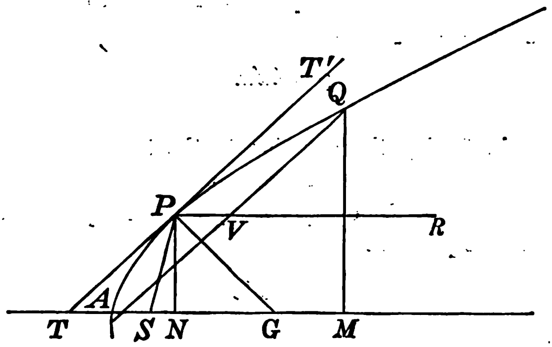
or

$$y = x \tan\theta + a \cot\theta,$$

the equation to the tangent which cuts the axis at an angle $\theta$.

Hence a diameter bisects all chords parallel to the tangent at its extremity.

**91.** *To find the equation to a parabola referred to a diameter and its tangent as axes.*



Let $Q$ be any point on the curve; $PR$, $PT'$ the new axes; let $PTA = \theta$.

Let the coordinates of $Q$ referred to the old axes be $x$, $y$, to the new $x'$, $y'$.

Then $$PN = NG \cot\theta = 2a\cot\theta,$$

$$AN = \frac{1}{2}AT = a\cot^2\theta,$$

$$\therefore x = AN + PV + VR = a\cot^2\theta + x' + y'\cos\theta,$$

$$y = QR + RM = QR + PN = y'\sin\theta + 2a\cot\theta,$$

$$\therefore (y'\sin\theta + 2a\cot\theta)^2 = 4a\,(x' + y'\cos\theta + a\cot^2\theta).$$

Expanding and reducing

$$y'^2 = \frac{4a}{\sin^2\theta}\,x',$$

the equation required.

The quantity $\frac{4a}{\sin^2\theta}$ is called the parameter of the diameter $PR$; it is equal to $4SP$, for $SP = a + AN = a + a\cot^2\theta = a\operatorname{cosec}^2\theta$.

Hence if $a'$ be the distance from the focus of any point on a parabola, the equation to the parabola referred to the diameter and tangent at that point is

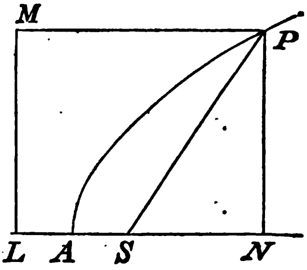
$$y^2 = 4a'x.$$

Since in finding the equation to the tangent we did not introduce the condition that the axes were rectangular, the proof holds equally good for oblique axes, and the equation to the tangent to the parabola

$$y^2 = 4a'x$$

is $$yy_1 = 2a'(x + x_1).$$

92. *Polar equation to the parabola.*



Let the focus be the pole, and the axis the initial line; then

$$SP = PM = LS + SN,$$

or $$r = 2a + r\cos\theta,$$

$$r = \frac{2a}{1 - \cos\theta} = a\operatorname{cosec}^2\frac{\theta}{2}.$$

If $\theta$ be measured from $A$, so that $ASP = \theta$,

$$r = \frac{2a}{1 + \cos\theta} = a\sec^2\frac{\theta}{2}.$$

**93.** *To find the polar equation to the tangent to a parabola.*

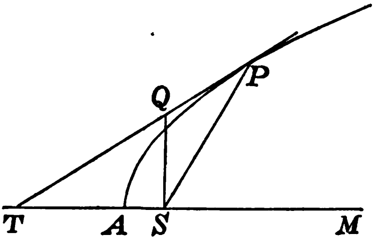
Let $Q$ be any point in the tangent $TQ$; then

$SQ \sin SQT = ST \sin QTS.$

But $ST = r_1$,

$QTS = QPS = \frac{\theta_1}{2}$,

$SQT = QSM - STQ = \theta - \frac{\theta_1}{2}$;



$$\therefore r \sin\left(\theta - \frac{\theta_1}{2}\right) = r_1 \sin\frac{\theta_1}{2} = 2a \operatorname{cosec}\frac{\theta_1}{2},$$

$$\therefore 2r \sin\left(\theta - \frac{\theta_1}{2}\right) \sin\frac{\theta_1}{2} = 4a,$$

$$\therefore r\{\cos(\theta - \theta_1) - \cos\theta\} = 2a,$$

the equation required.

We might have deduced the equation to the tangent from the consideration that it cuts the parabola in two coincident points; we will however do this hereafter, for any conic section, and consider the parabola as a particular case.

## EXAMPLE VIII.

1. If $P$ be any point on a parabola, $A$ the vertex and $PN$ drawn perpendicular to $AP$ cut the axis in $K$, then $NK$ is equal to the latus rectum.

2. If the distance of a point from the focus be equal to the latus rectum, then its abscissa is equal to $3a$.

3. Prove that the length of a side of an equilateral triangle inscribed in a parabola, so that one angle coincides with the vertex, is $8\sqrt{3}a$.

4. Find the length of the side of an equilateral triangle of which one angle coincides with the focus, and the others lie on the parabola $y^2 = 4ax$.

5. A double ordinate $PNP'$ of a parabola $= 8a$, prove that $AN = 4a$, and that $PAP'$ is a right angle.

6. If a circle pass through a fixed point and touch a fixed straight line, its centre lies on a parabola.

7. The diameter of the circle passing through the extremities of the latus rectum, and the vertex is five fourths of the latus rectum.

8. Two parabolas have the same axis and vertex, but the latus rectum of one is double that of the other; prove that any chord of the greater passing through the common vertex is bisected by the lesser parabola.

9. If $(x_1 y_1)$, $(x_2 y_2)$, be two points on the parabola $y^2 = 4ax$ and $x_2 - x_1 = a$, $y_1 y_2 = a^2$, determine $x_1$, $y_1$, $x_2$, $y_2$.

10. Prove that the area of a triangle described in the parabola $y^2 = 4ax$ is

$$\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1),$$

where $y_1$, $y_2$, $y_3$, are the ordinates of the angular points.

11. The abscissa and double ordinate of a segment of a parabola are $a$, $b$ and the diameters of the circumscribed and inscribed circles $D$, $d$; prove that $D + d = a + b$.

12. If a circle cut a parabola in 4 points, the algebraic sum of their ordinates is zero.

13. The circle described on a focal chord as diameter, touches the directrix.

14. The points of contact of two tangents being given, find their points of intersection.

15. Find that point on a parabola, the tangent at which cuts off equal intercepts from the axes.

16. If the tangent at $P$ meet the directrix at $D$, $DSP$ is a right angle.

17. A common tangent is drawn to a parabola, and the circle described in its latus rectum as diameter; prove that the angle between the lines drawn from the focus to the points of contact is bisected by the latus rectum.

18. The portion of the tangent at the vertex intercepted between the vertex and any diameter is bisected by the tangent at the extremity of that diameter.

19. If a circle pass through the vertex and focus, cutting the tangent at the vertex in $Q$, the tangent to the circle at $Q$ will touch the parabola.

20. If the tangent at $P$ cut the axes in $T$, $Y$, respectively,

$$TP \,.\, TY = TN \,.\, TS.$$

21. $T$ is any point, $TP$, $TQ$ the tangents through $T$, find the equations to $TP$, $TQ$, $SP$, $ST$, $SQ$, and prove that $TSP = TSQ$, and $QTS = TPS$.

22. The circle described on $SP$ as diameter touches the tangent at the vertex.

23. If $QQ'$ be a focal chord, $PV$ the diameter bisecting it, $SP = PV$ and $QV = 2SP$.

24. A straight line touches the curves

$$y^2 = 4ax, \text{ and } x^2 + y^2 = c^2,$$

find its equation.

25. Two tangents to the parabola $y^2 = 4ax$, are

$$y = x \tan \theta_1 + a \cot \theta_1,$$
$$y = x \tan \theta_2 + a \cot \theta_2,$$

find the locus of their intersection when $\cot \theta_1 + \cot \theta_2 = k$.

26. Find the locus, when $\cot \theta_1 - \cot \theta_2 = k$.

27. Find the locus, when $\tan \theta_1 . \tan \theta_2 = k$.

28. Find the locus, when $\sin \theta_1 . \sin \theta_2 = k$.

29. Find the distance of the vertex and focus from the tangent $y = x \tan \alpha + a \cot \alpha$.

30. If straight lines be drawn from the focus cutting the tangent $y = x \tan \theta + a \cot \theta$ at an angle $\alpha$, they will intersect it in the tangent

$$y = x \tan \alpha + a \cot \alpha.$$

31. Two normals to a parabola are always at right angles, find the locus of their intersection.

32. The point of intersection of the perpendicular from the focus on any normal to $y^2 = 4ax$, lies on the parabola $y^2 = a(x + a)$.

33. If $SPG$ be an equilateral triangle, $SP$ is equal to the latus rectum.

34. A circle touches a parabola at $A$, cuts it at $B$, $C$, and passes through a point $E$ on the axis; $BC$ cuts the axis in $D$; if $K$ be the middle point of $DE$, prove that $BK$ is the normal at $B$.

35. Find the equations to the common chords of the curves

$$y^2 = 4ax, \quad x^2 + y^2 = 2cx.$$

36. $BC$, $CD$ are two arcs of a parabola, such that the lengths of those parts of the diameters between their middle points $E$, $F$, and the curve are equal; prove that $EF$ is parallel to the tangent at $C$.

37. Two equal parabolas have the same axis, and a chord $Qq$ of the one making a constant angle with the axis cuts the other in $P$; prove that $PQ \cdot Pq$ is constant.

38. If there are three tangents to a parabola, the triangle formed by their intersections is half that whose angular points are the points of contact.

39. If a circle be described with centre $S$, and radius $SA$, and any focal chord $Pp$ cut this circle in $Q$, $q$, $PQ \cdot pq = a^2$.

40. Find the locus of the vertex of a parabola which has a given focus and touches a given straight line.

41. In the radius vector $SP$, $Q$ is taken equal to the semiordinate, find the locus of $Q$.

42. If $PSp$ be a focal chord, $l$ the semi-latus rectum,

$$\frac{1}{SP} + \frac{1}{Sp} = \frac{2}{l}.$$

43. If $r_1$, $r_2$ be two radii vectores at right angles to one another, $l$ the semi-latus rectum,

$$\left(\frac{1}{r_1} - \frac{1}{l}\right)^2 + \left(\frac{1}{r_2} - \frac{1}{l}\right)^2 = \frac{1}{l^2}.$$

44. If $PM$ be the perpendicular on the directrix from $P$, $SM^2 = 4a \cdot SP$.

45. If $Pp$ be a focal chord, prove that the triangle $PAp \propto (Pp)^{\frac{1}{2}}$.

46. If the tangent at $Q$ be parallel to the focal chord $Pp$, $SP \cdot Sp = 2l \cdot SQ$.

47. If $Q$ be on the focal chord $Pp$, and $SQ = Pp$, find the locus of $Q$.

48. $PQ$ is a chord through a given point $O$, cutting the parabola in $PQ$; prove that $PO \cdot OQ$ is least when $PQ$ is perpendicular to the axis.

49. The abscissæ of two points on a parabola are $x$, $3x$, and their focal distances $r$, $2r$; determine their positions.

50. Given a diameter and its tangent, find the locus of the vertex.

51. Find also the locus of the focus.

52. The directrix is the polar of the focus.

53. The poles of all straight lines through the foot of the directrix lie on the latus rectum.

54. The pole of any tangent to the parabola $y^2 = 4ax$, with respect to the circle $x^2 + y^2 = c^2$, lies on the parabola

$$ay^2 + c^2x = 0.$$

55. Two parabolas have their axes perpendicular to each other; prove that if they cut each other in four points, these points will lie on a circle.

56. Three parabolas, whose axes are parallel, intersect; prove that their chords of intersection pass through a single point.

57. The area and base of a triangle being given, find the locus of the intersection of perpendiculars from the extremities of the base on the opposite sides.

58. Find the locus of the centre of a circle inscribed in a sector of a given circle, one of the radii of the sector being fixed.

59. In the triangle $ABC$ if $\tan A \,.\, \tan \frac{B}{2} = k$, and $AB$ be fixed, find the locus of $C$.

60. $A$ is the origin, $B$ a point on the axis of $y$, $BQ$ a line parallel to the axis of $x$; in $AQ$, $P$ is taken such that its ordinate is equal to $BQ$: find the locus of $P$.

61. In the triangle $ABC$, $BC$ is given, and $\tan B + \tan C = k$; find the locus of $A$.

62. Find the locus of the centre of a circle which touches a given circle and given line.

63. $SY$ is the perpendicular from the focus of the tangent $PY$, find the locus of the centre of the circle circumscribing $SYP$.

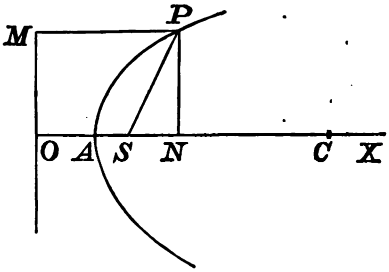
64. Find the locus of the focus of a parabola, which has a given vertex and touches a given line.

# CHAPTER VII.

## The Ellipse.

**94.** A conic section is the locus of a point, the distance of which from a given point is always in a given ratio to its distance from a given straight line; if this ratio be less than unity the curve is called an ellipse.

We proceed to find the equation to the ellipse.



Let $S$ be the focus, $OM$ the directrix, $SO$ perpendicular to $OM$; take $OS$, $OM$ as the axes of $x$ and $y$ respectively: let $P$ be any point on the curve, draw $PN$, $PM$ perpendicular to the axes, and join $SP$: let $OS = c$.

Join $SP$, let $OS = c$.

Then $SP = ePM = e \, . \, ON$;

$$\therefore \; SN^2 + PN^2 = e^2 \, . \, ON^2 \,;$$

or $$(x - c)^2 + y^2 = e^2x^2 \,;$$

or $$(1 - e^2)\, x^2 - 2cx + y^2 + c^2 = 0,$$

the equation required.

Change the origin to a point $C$ on $OS$ produced, such that $OC = h$, the directions of the axes being unaltered; then in the equation to the ellipse referred to the new axes, we must write $x + h$ for $x$;

$$\therefore\ (1-e^2)(x+h)^2 - 2c(x+h) + y^2 + c^2 = 0;$$

or

$$(1-e^2)x^2 + 2\{(1-e^2)h - c\}x + y^2 + (1-e^2)c^2 - 2ch + c^2 = 0.$$

In this equation put $h = \dfrac{c}{1-e^2}$;

then
$$(1-e^2)h^2 - 2ch + c^2 = \frac{c^2}{1-e^2}(1-2+1-e^2)$$
$$= -\frac{e^2c^2}{1-e^2};$$
$$\therefore\ (1-e^2)x^2 + y^2 - \frac{e^2c^2}{1-e^2} = 0;$$

or

$$\frac{x^2}{\dfrac{e^2c^2}{(1-e^2)^2}} + \frac{y^2}{\dfrac{e^2c^2}{1-e^2}} = 1.$$

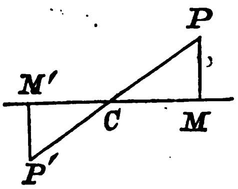
In this equation let
$$\frac{c^2}{(1-e^2)^2} = a^2, \quad \frac{c^2}{1-e^2} = b^2;$$
then the equation becomes
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

95. Definition. If there be a point such that all chords of a curve drawn through that point are bisected in it, that point is called the centre of the curve, and chords drawn through it are called diameters.

The origin is the centre, and chords through the origin diameters.

Let $P$ be a point on the curve, of which the coordinates are $x$, $y$, then since
$$\frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} = 1,$$

$(-x, -y)$ is also a point on the curve, that is, if $P'$ be the point in which $PC$ produced cuts the curve,



$$CM' = CM,\ P'M' = PM;$$

and $$\therefore\ CP' = CP.$$

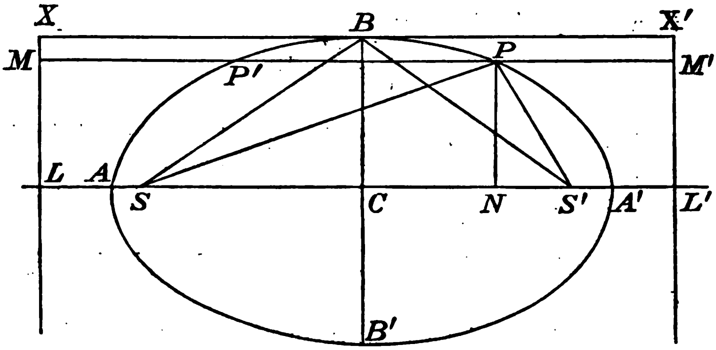
**96.** To examine the form of the curve represented by this equation.

Since the equation only involves the squares of $x$ and $y$, it will be unaltered if we substitute either $-x$, or $-y$, or both, for $x$ and $y$.

It is therefore symmetrical with respect to the axes of $x$ and $y$, that is, the axes divide it into four equal and similar portions.

Since the equation may be put into the form

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$



if $x^2 > a^2$, $y$ is impossible; similarly, if $y^2 > b^2$, $x$ is impossible; the curve has therefore no infinite branches and is entirely contained in the rectangle whose sides are the straight lines represented by the equations

$$x = \pm a,$$
$$y = \pm b.$$

In the equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

as $x$ increases from $0$ to $\pm a$, $y$ diminishes from $\pm b$ to $0$; the curve must therefore be of the form represented in the figure, where $CA = CA' = a$, $CB = CB' = b$;

$AA'$, $BB'$ are called the major and minor axes.

**97.** Since to every point $P$ on one side of the axis of $y$ there corresponds a point $P'$ at an equal distance on the other side; to the focus $S$ and directrix $LM$ there must correspond another focus $S'$ and directrix $L'M'$ at equal distances from $C$.

**98.** Let $P$ be any point on the curve, draw $MPM'$ perpendicular to the directrices, join $SP$, $S'P$.

Then $$SP = e\,.\,PM,\quad S'P = e\,.\,PM';$$
$$\therefore\ SP + S'P = e\,.\,LL'.$$

But $$SA = e\,.\,AL,\ SA' = e\,.\,A'L,\ \text{and } AL = AL';$$
$$\therefore\ AA' = e\,.\,LL,$$
$$\therefore\ SP + S'P = AA' = 2a,$$

or the sum of the focal distances of any point is equal to the axis-major.

Since $$SP = e\,.\,PM = e\,.\,LN,$$

and $$LN = CL + CN = \frac{a}{e} + x,$$

$$SP = a + ex.$$

Similarly, $S'P = a - ex,$

$$\therefore\ SB = a.$$

Now $BC^2 = (1 - e^2)\,a^2,$

$$\therefore\ SC^2 = a^2 - (1 - e^2)\,a^2 = e^2a^2.$$

### 99. *The auxiliary circle.*

The equation to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

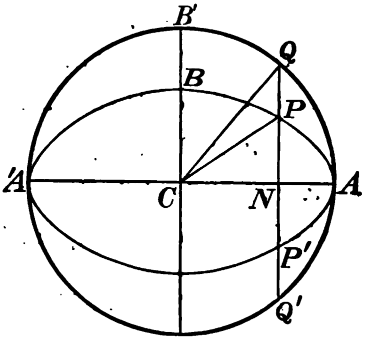
may be put in the form

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

Here for any value of $x$ we have two equal and opposite values of $y$.

Let $AQB'$ be the circle described on the axis major as diameter, then the equation to this circle is

$$x^2 + y^2 = a^2,$$



or $y^2 = a^2 - x^2,$

$$\therefore\ y = \pm(a^2 - x^2)^{\frac{1}{2}}.$$

Let $CN = x$: divide $QN$ in $P$ so that $PN = \frac{b}{a} QN$.

Now $$QN = (a^2 - x^2)^{\frac{1}{2}};$$

$$\therefore\ PN = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}};$$

that is, $P$ is a point on the ellipse.

Hence if on the axis-major of an ellipse as diameter a circle be described, the ellipse will cut all its ordinates in the constant ratio

$$\frac{PN}{QN} = \frac{b}{a}.$$

This circle is called the auxiliary circle.

**100.** Since the equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and since $\cos^2\phi + \sin^2\phi = 1$, for all values of $\phi$,

$$\therefore \text{ if } x = a\cos\phi,\ y = b\sin\phi.$$

Let us consider the geometrical meaning of the angle $\phi$.

Let $P$ be any point $(x, y)$ on the ellipse, $Q$ the corresponding point on the auxiliary circle, then

$$QN = \frac{a}{b};\ \ PN = \frac{a}{b}y;$$

also $$CN = x;$$

but $$CN = CQ\cos QCN = a\cos QCN,$$

$$PN = \frac{b}{a},\ QN = \frac{b}{a}a\sin QCN = b\sin QCN;$$

therefore if $P$ be any point on an ellipse, and $QCN = \phi$, the coordinates of $P$ are

$$a\cos\phi, \quad b\sin\phi.$$

$QCN$ is called the *excentric angle.*

Many problems on the ellipse may be easily solved by using coordinates in terms of the excentric angle.

### 101. *Polar equation to the ellipse.*

We have already found the equation to a conic section when the focus is pole to be

$$r = \frac{l}{1 - e\cos\theta}.$$

Let us find the equation to the ellipse when the centre is pole.

In the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

write $r\cos\theta$, $r\sin\theta$ for $x$ and $y$ respectively,

$$\therefore\ r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1;$$

$$\therefore\ r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}:$$

but $$b^2 = a^2(1 - e^2),$$

$$\therefore\ r^2 = \frac{a^2(1 - e^2)}{(1 - e^2)\cos^2\theta + \sin^2\theta}$$

$$= \frac{a^2(1 - e^2)}{1 - e^2\cos^2\theta}.$$

In this equation when $\theta = 0$, $r = a$, and as $\theta$ increases from 0 to $\frac{\pi}{2}$, $r$ diminishes from $a$ to $b$.

Therefore the radius vector of an ellipse always lies between $a$ and $b$.

**102.** *The latus rectum*, $2l$, $=\dfrac{2b^2}{a}$.

Putting $x = ae$ in the equation to the ellipse,

$$e^2 + \frac{l^2}{b^2} = 1, \quad l^2 = (1-e^2)\,b^2 = \frac{b^4}{a^2};$$

$$\therefore\ l = \frac{b^2}{a}.$$

## Example IX.

1. If the vertex be origin the equation in Art. 94 becomes

$$(1-e^2)\,x^2 + y^2 = ecx.$$

2. If the further extremity of the axis-major be origin, the same equation becomes

$$(1-e^2)\,x^2 + y^2 + ecx = 0.$$

3. If the focus be origin, the same equation becomes

$$(1-e^2)\,x^2 - 2e^2cx + y^2 = e^2c^2.$$

4. If the origin be transferred to any point $(h, k)$, the direction of the axes being unaltered, the equation to the ellipse becomes

$$\frac{(x+h)^2}{a^2} + \frac{(y+k)^2}{b^2} = 1.$$

5. If the origin be the lower extremity of the axis minor, the equation becomes

$$\frac{x^2}{a^2} = \frac{(2b-y)\,y}{b^2}.$$

6. If the point $(h, k)$ be on the curve, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2ha}{a^2} + \frac{2ky}{b^2} = 0.$$

7. Determine the excentricities and latera recta of the following ellipses:

(i) $x^2 + 2y^2 = 2$; (ii) $3x^2 + 4y^2 = 12$;
(iii) $2x^2 + y^2 = 1$; (iv) $ax^2 + by^2 = ab$;
(v) $(a - b) x^2 + (a + b) y^2 = a^2 - b^2$;
(vi) $2abx^2 + (a^2 + b^2) y^2 = (a + b)^2$.

8. Determine the distances between the foci in the ellipses in question 7.

9. If $S$, $S'$ be the foci, $B$ the extremity of the axis minor, and $SBS'$ be a right angle, determine the excentricity.

10. If $KSK'$ be the latus rectum, and $KCK'$ be an equilateral triangle, determine the excentricity.

11. Determine the excentric angle of the point $(h, k)$.

12. Determine the excentric angle when the abscissa is equal to the semi-ordinate.

13. Determine the excentric angle for the extremity of the latus rectum.

14. Prove that the equation to the chord of the points whose excentric angles are $\phi_1$, $\phi_2$ is

$$\frac{x}{a}\cos\frac{\phi + \phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1 + \phi_2}{2} = \cos(\phi_1 - \phi_2).$$

15. From Question 14 find the equation to the tangent at the point whose excentric angle is $\phi$.

16. The square on a diameter is an arithmetic mean between the squares on the axes; find its inclination to the axis major.

17. A diameter is a geometric mean between the axes; find its inclination to the axis major.

18. The square on a diameter is a harmonic mean between the squares on the axes; find its inclination to the axis major.

19. The axis minor is a geometric mean between the axis major and latus rectum.

20. The length of a diameter, its inclination to the axis, and the excentricity are known; determine the axes.

21. The length of a diameter is known and its inclination to the axis minor, the length of which is also known; determine the axis major.

22. Determine the inclination and excentric angle of the semidiameter whose length is $\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$.

23. If the excentric angle of a diameter be $\phi$, its length is

$$2\,(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}.$$

24. If $(r_1\theta_1)$ $(r_2\theta_2)$ be two points on an ellipse,

$$\frac{1}{r_1^2} - \frac{1}{r_2^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\sin^2\theta_1 - \sin^2\theta_2).$$

25. The point $(c, c)$ lies on the ellipse, determine $c$.

26. The point $lz, mz$, lies on the ellipse, determine $z$.

27. The point $a - z, b - z$, lies on the ellipse, determine $z$.

### TANGENT AND NORMAL TO AN ELLIPSE.

**103.** Let $(x_1y_1)$, $(x_2y_2)$, be two points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

then
$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

is the equation to the secant passing through these points.

But
$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1=\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2};$$

$$\therefore\ \frac{y_2^2-y_1^2}{b^2}=\frac{x_1^2-x_2^2}{a^2},\ \text{or}\ \frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}.$$

The equation to the secant becomes therefore

$$\frac{(y-y_1)(y_2+y_1)}{b^2}+\frac{(x-x_1)(x_2+x_1)}{a^2}=0.$$

In this equation let $x_2=x_1$, $y_2=y_1$, then the secant becomes the tangent and the equation becomes

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

The equation to the tangent at the point $(x_1y_1)$ is therefore

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

**104.** Equation to the tangent in terms of the excentric angle.

Since $(x_1y_1)$ is a point on the ellipse, we may put

$$x_1=a\cos\phi_1,\quad y_1=b\sin\phi_1,$$

where $\phi_1$ is the excentric angle of the point $(x_1 y_1)$; therefore the above equation to the tangent becomes

$$\frac{x}{a}\cos\phi_1 + \frac{y}{b}\sin\phi_1 = 1.$$

**105.** Perpendicular on the tangent from the origin.

By Art. 33, the perpendicular on the line

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

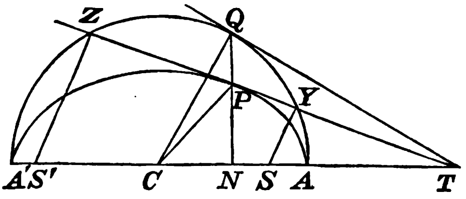
is

$$\frac{1}{\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right)^{\frac{1}{2}}}, \text{ or } \frac{a^2b^2}{(b^4x_1^2 + a^4y_1^2)^{\frac{1}{2}}}.$$

Similarly the perpendicular in terms of the excentric angle is

$$\frac{ab}{(b^2\cos^2\phi + a^2\sin^2\phi)^{\frac{1}{2}}}, \text{ or } \frac{b}{(1 - e^2\cos^2\phi)^{\frac{1}{2}}}.$$

**106.** Let $P$ be any point $(x_1, y_1)$ on the ellipse, let $NP$ be produced to meet the auxiliary circle in $Q$, then the coordinates of $Q$ are $x_1$, $y_1'$, where $y_1' = \frac{a}{b}y_1$.



Now the equation to the tangent to the ellipse at $P$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

That to the tangent to the circle at $Q$ is

$$xx_1 + yy_1' = a^2.$$

In each of these equations put $y=0$.

Then the lines represented by them cut the axis of $x$ in the same point $T$, such that $CT=\dfrac{a^2}{x_1}$.

Hence, if we wish to draw a tangent to an ellipse at any point $P$, draw the tangent $QT$ to the auxiliary circle, cutting the axis in $T$, and join $PT$, $PT$ is the tangent required.

**107.** *To find the perpendicular from the focus on the tangent.*

Let the equation to the tangent be

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1.$$

Then the perpendicular from $(ae, 0)$ on this line is

$$\frac{1-e\cos\phi}{\left(\dfrac{\cos^2\phi}{a^2}+\dfrac{\sin^2\phi}{b^2}\right)^{\frac{1}{2}}},$$

or $$\frac{a(1-e\cos\phi)(1-e^2)^{\frac{1}{2}}}{(1-e^2\cos^2\phi)^{\frac{1}{2}}}, \text{ since } b^2=a^2(1-e^2).$$

Similarly the perpendicular $S'Z$ on $TP$ from the other focus $S'$ is

$$\frac{a(1+e\cos\phi)(1-e^2)^{\frac{1}{2}}}{(1-e^2\cos^2\phi)^{\frac{1}{2}}}.$$

Hence $$SY.S'Z=a^2(1-e^2)=b^2,$$

or $$SY.S'Z=BC^2.$$

**108.** *To prove that $Y$, $Z$ lie on the auxiliary circle.*

The equation to $TP$ is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1,$$

and that to $SY$ is therefore

$$\frac{x-ae}{b}.\sin\phi - \frac{y\cos\phi}{ba} = 0,$$

or

$$\frac{x\sin\phi}{b} - \frac{y\cos\phi}{a} = \frac{ae}{b}\sin\phi.$$

Square both sides of each of these equations and add,

$$\therefore\ (x^2+y^2)\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right) = 1 + \frac{a^2e^2\sin^2\phi}{b^2}$$

$$= a^2\left(\frac{1}{a^2}+\frac{e^2\sin^2\phi}{b^2}\right)$$

$$= a^2\left(\frac{\cos^2\phi+\sin^2\phi}{a^2}+\frac{e^2\sin^2\phi}{b^2}\right)$$

$$= a^2\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right),$$

since $b^2 = a^2(1-e^2)$,

$\therefore\ x^2+y^2 = a^2$, that is, $Y$ lies on the auxiliary circle.

**109.** To find the condition that the line

$$x\cos\theta + y\sin\theta = p$$

may touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b} = 1.$$

We might proceed by finding the conditions that the given equation should coincide with that to the tangent at the point $(x_1y_1)$, and then eliminate $x_1$, $y_1$ between the equations of condition thus found and the equation to the ellipse; we will, however, find the condition that the line

$$x\cos\theta + y\sin\theta = p$$

shall meet the ellipse in two coincident points.

Since $$x\cos\theta + y\sin\theta = p,$$

$$y = \frac{p - x\cos\theta}{\sin\theta}.$$

Substituting in the equation to the ellipse and multiplying by $\sin^2\theta$, we get

$$\frac{x^2}{a^2}\sin^2\theta + \frac{(p - x\cos\theta)^2}{b^2} = \sin^2\theta,$$

or $$x^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right) - \frac{2px}{b^2}\cos\theta + \frac{p^2 - b^2\sin^2\theta}{b^2} = 0.$$

Now if the roots of this equation are equal, the straight line must be a tangent; but if the equation

$$Ax^2 + Bx + C = 0$$

have equal roots, we know from algebra that $B^2 = 4AC$.

Hence we must have

$$\frac{p^2\cos^2\theta}{b^4} = \frac{p^2 - b^2\sin^2\theta}{b^2}\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b}\right);$$

$$\therefore\ \frac{p^2\sin^2\theta}{a^2} = b^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\sin^2\theta,$$

or $$p^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

and the equation to the straight line becomes

$$x\cos\theta + y\sin\theta = \pm(a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}.$$

In this equation it must be remembered that $\theta$ is *not* the excentric angle of the point of contact, but the angle which the perpendicular from the origin makes with the axis of $x$.

**110.** Equation to the normal at $(x_1y_1)$.

Since the line passes through the point $x_1y_1$, its equation must be of the form

$$A(x - x_1) + B(y - y_1) = 0;$$

also, since it is perpendicular to the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

$$\frac{Ax_1}{a^2}+\frac{By_1}{b^2}=0.$$

The equation therefore becomes

$$\frac{x_1y}{a^2}-\frac{x_1y_1}{a^2}=\frac{y_1x}{b^2}-\frac{x_1y_1}{b^2},\quad\text{or}\quad \frac{y-y_1}{a^2y_1}=\frac{x-x_1}{b^2x_1}.$$

**111.** If we put $x_1=a\cos\phi_1$, $y_1=b\cos\phi_1$, the equation

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

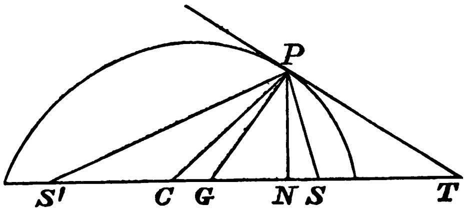
becomes

$$y-b\sin\phi_1=\frac{a}{b}\tan\phi_1(x-a\cos\phi_1),$$

or $$ax\sin\phi_1-by\cos\phi_1=(a^2-b^2)\sin\phi_1\cos\phi_1.$$

**112.** In this equation put $y=0$;

$$\therefore\ x=\frac{a^2-b^2}{a^2}\cos\phi_1=ae^2\cos\phi_1=e^2x_1.$$



If therefore $PG$ be the normal at $P$, cutting the axis at $G$, $CG=e^2CN$.

**113.** The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal distances.

We have
$$SP = a - ex_1,$$
$$SP' = a + ex_1.$$

Now
$$SG = SC = CG = ae - e^2x_1,$$
$$S'G = SC + CG = ae + e^2x_1;$$
$$\therefore\ SG : S'G :: SP : S'P;$$
$$\therefore\ PG \text{ bisects the angle } S'PS.$$

If the normal bisects the internal angle, it is evident that the tangent must bisect the external angle between the same straight lines.

**114.** By a proof precisely similar to that of Arts. 58, 59, we can shew that when $(x_1y_1)$ is a point without the ellipse, the equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

represents the chord of contact of tangents through $(x_1y_1)$.

This equation represents the polar of $(x_1y_1)$ wherever $(x_1y_1)$ may be; and it may be proved, as in the case of the circle and parabola, that the chords of contact of tangents drawn from any point on the line always pass through the point.

## EXAMPLE X.

Find the equations to tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which fulfil the following conditions,

1. Make an angle $\theta$ with the axis of $x$;

2. Cut off a given triangle between the axes;

3. Are parallel to

$$\frac{x}{a} + \frac{y}{b} = 1;$$

4. Make equal angles with the axes;

5. Are at a given distance from the point $(h, k)$.

6. Find the equation to the tangent $TPt$, cutting the axes in $T$, $t$, respectively, when

$$CT + Ct = l.$$

7. Find the equation to the tangent $TPt$, when the perimeter of the triangle $TPt$ is equal to $2s$.

8. Find the equation to the tangent, when the perpendiculars from the foci are in a given ratio.

9. Find the conditions that the lines

$$Ax + By + C = 0,$$

and

$$y = mx + c,$$

should touch the ellipse.

10. If $TPt$ be a tangent, cutting the axis minor at $t$, and $CTt = \omega$, find the point of contact.

11. Find the equation to the tangent at the extremity of the latus rectum of the ellipse

$$x^2+2y^2=3.$$

12. Find the equation to the tangent which is parallel to *SB*.

13. The equation to the diameter drawn perpendicular to the tangent at $(x_1y_1)$, is

$$\frac{xy_1}{b^2}=\frac{x_1y}{a^2}.$$

14. Prove that the locus of the point of intersection of this line with the line $x=x_1$, is the ellipse

$$\frac{x^2}{a^2}+\frac{b^2y^2}{a^4}=1.$$

15. If $h$, $k$, be the intercepts on the axes of any tangent,

$$\frac{a^2}{h^2}+\frac{b^2}{k^2}=1.$$

16. Two tangents are such that the product of the tangents of the angles they make with the axis of $x$ is $m$, prove that they intersect on the curve

$$y^2-b^2=m^2(x^2-a^2).$$

17. Two tangents intersect at right angles, prove that their point of intersection lies on the circle

$$x^2+y^2=a^2+b^2.$$

18. If *PG* be the normal at *P*, cutting the axis at *G*, and *CK* be perpendicular to it, determine *PG*, and *CK* in terms of $\phi$, the excentric angle of *P*.

19. If the normal cut the curve again at *Q*, find *PQ*.

20. Find the equations to the normals which pass through the points $c$, $0$; $0$, $c$; $h$, $k$; respectively.

21. If $PG$ be the normal at $P$, then

$$2\tan CPG = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin 2\phi.$$

22. If $PG$ cut the minor axis in $g$, find $Gg$.

23. $Q$ is a point on the tangent at the extremity of the axis minor, $QP$ the tangent at $P$; if $BQ = \sqrt{3}a$, then

$$PQ = BP.$$

24. $PR$, $QR$ are two normals at $P$, $Q$, cutting at right angles in $R$, $CRH$ is a semi-diameter; prove that the tangent at $H$ is parallel to $PQ$.

25. If $r$ be a radius $CP$ of an ellipse and $p$ the perpendicular on the tangent at $P$, then

$$p^2 = \frac{a^2b^2}{a^2 + b^2 - r^2}.$$

26. If $\theta$, $\psi$, be the angles made with the axis by the radius vector, and normal at any point of an ellipse, $\phi$ the excentric angle at that point,

$$\tan\theta\tan\psi = \tan^2\phi.$$

## DIAMETERS. CONJUGATE DIAMETERS.

**115.** *To find the locus of the middle points of the chords of an ellipse which are parallel to a fixed straight line.*

Some tangent to the ellipse must be parallel to the fixed straight line, and therefore the chords must be parallel to that tangent.

Let $$\frac{xx_1}{a^2} + \frac{yy}{b^2} = 1$$

be the equation to the tangent:

$$\text{then, } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = k$$

will be the equation to any chord.

To find where this chord cuts the ellipse:

$$\left(k - \frac{xx_1}{a^2}\right)^2 = \frac{y^2 y_1^{\ 2}}{b^4} = \left(1 - \frac{x^2}{a^2}\right)\frac{y_1^{\ 2}}{b^2};$$

$$\therefore \frac{x^2}{a^2}\left(\frac{x_1^{\ 2}}{a^2} + \frac{y_1^{\ 2}}{b^2}\right) - 2k\frac{xx_1}{a^2} = \frac{y_1^{\ 2}}{b^2} - k^2,$$

$$\text{or } \frac{x^2}{a^2} - 2k\frac{xx_1}{a^2} - \frac{y_1^{\ 2}}{b^2} + k^2 = 0;$$

if therefore $x'$, $y'$, be the coordinates of the middle point of this chord, $x'$ is equal to half the sum of the roots of this equation,

or $$x' = kx_1.$$

Similarly, $$y' = ky_1,$$

and therefore $$\frac{x'}{x_1} = \frac{y'}{y_1};$$

that is, $(x', y')$ is a point on the diameter through $(x_1, y_1)$.

The diameter parallel to the bisected chords is said to be conjugate to that bisecting them.

Its equation is evidently

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0.$$

**116.** Putting $x_1 = a\cos\phi_1$, $y = b\sin\phi_1$, the equation to the original diameter becomes

$$\frac{y}{b\sin\phi_1} = \frac{x}{a\cos\phi_1}$$

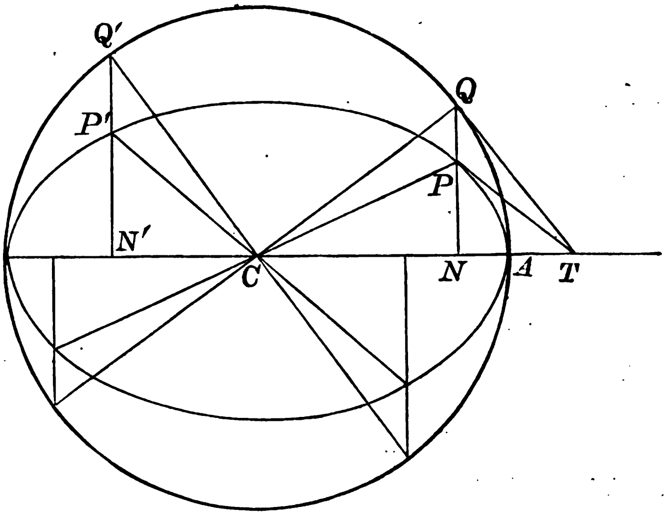
to the conjugate diameter

$$\frac{x\cos\phi_1}{a} + \frac{y\sin\phi_1}{b} = 0.$$

Let $CP$ be a diameter, $CP'$ its conjugate; then the equation to $CP$ may be written

$$\frac{x\sin\phi}{a} - \frac{y\cos\phi}{b} = 0,$$

where $\phi = QCA$.



Now the equation

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 0,$$

to $CP'$, may be written

$$\frac{x \sin\left(\phi + \frac{\pi}{2}\right)}{a} - \frac{y \cos\left(\phi + \frac{\pi}{2}\right)}{b} = 0.$$

The diameter conjugate to that whose excentric angle is $\phi$ is therefore that diameter whose excentric angle is

$$\frac{\pi}{2} + \phi.$$

Let us now find the diameter conjugate to $CP'$.

Its equation must be

$$\frac{x \sin(\phi + \pi)}{a} - \frac{y \sin(\phi + \pi)}{b} = 0,$$

or

$$\frac{x \sin\phi}{a} - \frac{y \cos\phi}{b} = 0.$$

But this is the equation to $CP$: if therefore $CP'$ is conjugate to $CP$, $CP$ is conjugate to $CP'$.

**117.** We might prove the property thus: the line

$$\frac{x \cos\phi}{a} + \frac{y \sin\phi}{b} = 0,$$

intersects the ellipse in the points

$$-a \sin\phi,\ b \cos\phi;\quad a \sin\phi,\ -b \cos\phi.$$

The equation to the tangent at the former of these points is

$$-\frac{x \sin\phi}{a} + \frac{y \cos\phi}{b} = 1,$$

which is parallel to

$$-\frac{x \sin\phi}{a} + \frac{y \cos\phi}{b} = 0;$$

$CP'$ therefore bisects all chords parallel to $CP$.

**118.** If $\theta$, $\theta'$, be the angles $PCA$, $P'CA$, respectively; $\phi$, $\phi'$ the excentric angles $QCA$, $Q'CA$,

$$\tan\theta = \frac{b}{a}\tan\phi\,; \quad \tan\theta' = \frac{b}{a}\tan\phi'.$$

But
$$Q'CA = \frac{\pi}{2} + QCA,$$

$$\therefore \tan\phi\tan\phi' = -1,$$

$$\therefore \tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

**119.** If the coordinates of the extremity of a diameter be $x_1y_1$, those of the other extremity must be $-x_1 -y_1$.

The equation to the tangent at this extremity is therefore

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = -1,$$

which is parallel to the tangent

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

**120.** Let $a'$, $b'$, be the lengths of the conjugate semi-diameters $CP$, $CP'$; $\phi$ the angle $QCA$.

Then $\quad a'^2 = CP^2 = a^2\cos^2\phi + b^2\sin^2\phi.$

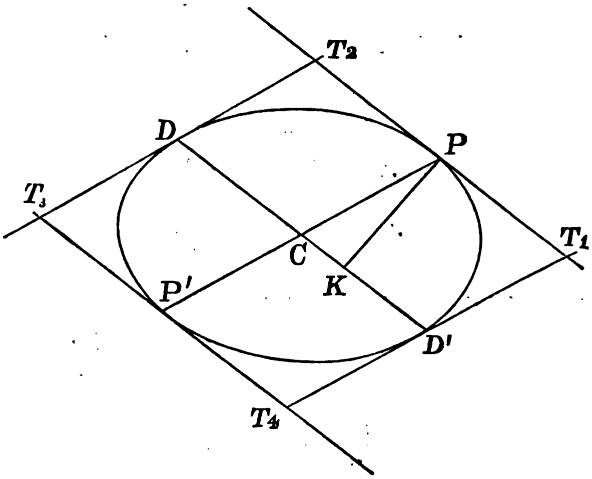
Also $\quad b'^2 = CP'^2 = CN'^2 + P'N'^2 = a^2\sin^2\phi + b^2\cos^2\phi\,;$

$$\therefore a'^2 + b'^2 = a^2 + b^2,$$

or the sum of the squares of the semi-diameters is constant.

**121.** *The area of the parallelogram circumscribing the ellipse whose sides are parallel to conjugate diameters is constant.*

Let $T_1T_2T_3T_4$ be such a parallelogram, $PCP'$, $DCD'$ conjugate diameters; draw the normal $PK$.



Let $\phi$ be the excentric angle of $P$. Then the parallelogram

$$T_1T_2T_3T_4 = 4\,PCDT_2 = 4CD\,.\,PK.$$

But
$$CD = (a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}};$$

$$PK = \frac{ab}{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}}} \text{ (Art. 105)},$$

$$\therefore\ CD\,.\,PK = ab,$$

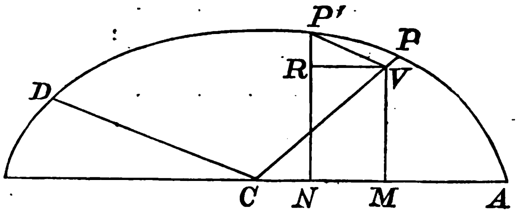
therefore the parallelogram required $= 4ab$.

**122.** If $\alpha$ be the angle between two conjugate diameters whose lengths are $a'$, $b'$, respectively, since the parallelogram contained by them is equal to $a'b'\sin\alpha$, we have

$$a'b'\sin\alpha = ab,$$

$$\text{or } \sin\alpha = \frac{ab}{a'b'}.$$

**123.** *To find the equation to the ellipse referred to a pair of conjugate diameters as axes.*



Let $CP$, $CD$ be the new axes, $\theta$, $\theta'$ the angles they respectively make with the axis of $x$.

Let $P'$ be any point on the ellipse; $(x, y)$, $(x', y')$ its coordinates referred to the old and new axes respectively.

Draw $VR$ parallel to $CA$, then

$$x = CN = CM - RV = CV \cos PCA + P'V \cos DCA$$

$$= x' \cos\theta + y' \cos\theta';$$

similarly,
$$y = x' \sin\theta + y' \sin\theta'.$$

Substitute in the equation to the ellipse; thus

$$\frac{(x'\cos\theta + y'\cos\theta')^2}{a^2} + \frac{(x'\sin\theta + y'\sin\theta')^2}{b^2} = 1,$$

$$\text{or } x'^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + 2x'y'\left(\frac{\cos\theta\cos\theta'}{a^2} + \frac{\sin\theta\sin\theta'}{b^2}\right)$$

$$+ y'^2\left(\frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2}\right) = 1,$$

Now the polar equation to the ellipse is

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta},$$

$$\text{or } \frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2};$$

$$\therefore \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{CP^2} = \frac{1}{a'^2}:$$

similarly,
$$\frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2} = \frac{1}{CD^2} = \frac{1}{b'^2}.$$

Also $\tan\theta\tan\theta' = -\frac{b^2}{a^2}$, since the diameters are conjugate;

$$\therefore \frac{\cos\theta\cos\theta'}{a^2} + \frac{\sin\theta\sin\theta'}{b^2} = 0;$$

therefore the above equation becomes

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1.$$

Hence we see that, whatever be the angle between the axes, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an ellipse referred to two conjugate diameters whose lengths are $2a$ and $2b$.

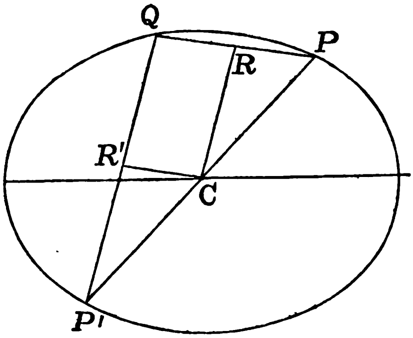
**124.** Since in Art. 103, no mention was made of the axes being rectangular, the equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

to the tangent at the point $(x_1\ y_1)$, will hold good generally provided the ellipse be referred to two conjugate diameters.

### 125. *Supplemental chords.*

Let $PCP'$ be any diameter of an ellipse, $Q$ any point on the curve: join $QP$, $QP'$; these are called supplemental chords.



The diameters parallel to a pair of supplemental chords are conjugate.

Bisect $QP$, $QP'$, in $R$, $R'$, respectively; join $CR . CR'$. Then we know that $CR$ bisects all chords parallel to $PQ$, and $CR'$ all chords parallel to $P'Q$.

But $$QR = \frac{1}{2}QP,\ PC = \frac{1}{2}PP',$$

therefore $CR$ is parallel to $P'Q$.

Similarly $CR'$ is parallel to $PQ$,
therefore $CR$, $CR'$, are conjugate diameters.

**126.** Hence if we wish to draw two conjugate diameters of an ellipse containing a given angle; take any diameter of the ellipse and on it describe a segment of a circle containing the required angle, let this segment cut the ellipse in $Q$, then the diameters parallel to $PQ$, $P'Q$ are conjugate, and contain the required angle.

## EXAMPLE XI.

1. WRITE down the equations to the diameters respectively conjugate to the following lines,

$$y = x;\ x + y = 0,\ ax = by;\ ay = bx;\ y \cos\theta = x \sin\theta;$$

$$\frac{x}{a} + \frac{y}{b} = 0.$$

2. The lengths of the equal conjugate diameters are $2(a^2+b^2)^{\frac{1}{2}}$, and their excentric angles $\frac{\pi}{4}, \frac{3\pi}{4}$, respectively.

3. The length of a semidiameter is $c$, find its excentric angle, and the equation to its conjugate.

4. If the tangent at the vertex $A$ cut any two conjugate diameters in $T$, $t$, then $AT \cdot At = b^2$.

5. The length of a semidiameter is $\frac{a+b}{2}$, find the equation to its conjugate.

6. If $\alpha$ be the angle between two conjugate diameters, which make angles $\theta$, $\theta'$, with the axis major,

$$\cos\alpha = e^2 \cos\theta \cos\theta'.$$

7. The angle between the equal conjugate diameters being $\frac{\pi}{3}$, shew that the excentricity is $\frac{\sqrt{6}}{3}$.

8. The equation to an ellipse being $2x^2 + 3y^2 = 4$, the diameters

$$y = 2x,\ x + 3y = 0,$$

are conjugate.

9. If $CP$, $CD$ be conjugate, and ordinates through $P$, $D$, meet another ellipse described on the same axis major, in $Q$, $E$, respectively, $CQ$, $CE$ are conjugate diameters of the second ellipse.

10. The locus of the middle points of chords joining the extremities of conjugate diameters is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

11. If $a'$, $b'$, be the lengths of $CP$, $CD$, and the angles $PCA$, $DCB$ be $\alpha$, $\beta$, respectively, then

$$\frac{a'^2-b'^2}{a^2-b^2}=\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}.$$

12. $S$ is a focus, $CP$, $CD$ conjugate diameters; the distance of $P$ from the diameter which is parallel to $SD$ is equal to $b$.

13. If $SP$ intersect $CD$ in $Q$, then $PQ=a$.

14. If $a'$, $b'$ be the lengths of $CP$, $CD$, and $\theta_1$, $\theta_2$ the angles they make with the axis of $x$, then

$$a'^2\sin 2\theta_1+b'^2\sin 2\theta_2=0.$$

15. The normals at $P$ and $D$ meet the major axis in $G$, $G'$, respectively, prove that $PG^2+DG'^2=\frac{b^2}{a^2}(a^2+b^2)$.

16. If $T$ be a point $(h, k)$ on the tangent at $P$, $l$ the length of $TP$, and $2b'$ of the diameter conjugate to $CP$,

$$l^2=\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)b'^2.$$

17. Tangents to an ellipse are drawn of lengths equal to $n$ times the conjugate semidiameters at their extremities, prove that the locus of their other extremities is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+n^2.$$

18. If $a'$, $b'$, be the lengths of two semidiameters at right angles to each other,

$$\frac{1}{a'^2}+\frac{1}{b'^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

19. If $\theta$ be the angle which these diameters make respectively with the major and minor axes,

$$\cos 2\theta=\frac{\dfrac{1}{b'^2}-\dfrac{1}{a'^2}}{\dfrac{1}{b^2}-\dfrac{1}{a^2}}.$$

20. If the ellipse be referred to these diameters as axes of coordinates, its equation will become

$$\frac{x^2}{a'^2}+2xy\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)\tan 2\theta+\frac{y^2}{b'^2}=1.$$

21. If $2c$ be the length of the equal conjugate semidiameters, the equation

$$x^2+y^2=c^2$$

represents the ellipse referred to these as axes.

22. If $\alpha$ be the acute angle between the axes of coordinates, the semiaxes of the ellipse, $x^2+y^2=c^2$, are

$$2c\cos\frac{\alpha}{2},\text{ and }2c\sin\frac{\alpha}{2}.$$

23. If $e$ be the excentricity of the same ellipse,

$$e=\left(\frac{2\cos\alpha}{1+\cos\alpha}\right)^{\frac{1}{2}}.$$

24. The equation to the auxiliary circle is

$$x^2+y^2+2(xy-c^2)\cos\alpha=2c^2.$$

# CHAPTER VIII.

## THE HYPERBOLA.

**127.** WE have already discussed the nature of the curves traced out by a point whose distance from the focus is equal to its distance from the directrix, or less than that distance in a given ratio: we will now consider the nature of the curve when the distance from the focus is greater than the distance from the directrix, and in a given ratio to it.

This curve is called a hyperbola.

**128.** *To find the equation to the hyperbola.*

Take the same axes as in Art. 94.

Since in that Article we did not introduce the condition that $e < 1$, the same equation

$$(1 - e^2)\, x^2 - 2cx + y^2 + c^2 = 0,$$

will represent this curve; or since $e > 1$ we may now write it

$$(e^2 - 1)\, x^2 + 2cx - y^2 - c^2 = 0.$$

Again, as in Art. 94, put $x + h$ for $x$.

Then as before,

$$(e^2 - 1)\, x^2 + 2\,\{(e^2 - 1)\, h + c\}\, x - y^2 + (e^2 - 1)\, h^2 + 2ch - c^2 = 0.$$

Let $$h = \frac{-c}{e^2-1}.$$

Then $$(e^2-1)h^2 + 2ch - c^2 = \frac{c^2}{e^2-1}\{1-2-(e^2-1)\}$$
$$= -\frac{e^2c^2}{e^2-1}.$$

And the equation becomes
$$(e^2-1)x^2 - y^2 - \frac{e^2c^2}{e^2-1} = 0,$$
$$\text{or } \frac{x^2}{\dfrac{e^2c^2}{(e^2-1)^2}} - \frac{y^2}{\dfrac{e^2c^2}{e^2-1}} = 1.$$

In this equation if we put
$$\frac{e^2c^2}{(e^2-1)^2} = a^2, \quad \frac{e^2c^2}{e^2-1} = b^2,$$
we obtain
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Here $$e^2 - 1 = \frac{b^2}{a^2}, \text{ or } e^2 = \frac{a^2+b^2}{a^2}.$$

**129.** Since the equation to the hyperbola may be derived from that to the ellipse by changing $b^2$ into $-b^2$, if we write $-b^2$ for $b^2$ in any equation or expression which represents a line or property of the ellipse, we shall obtain the equation or expression for the corresponding line or property of the hyperbola.

**130.** To examine the form of the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

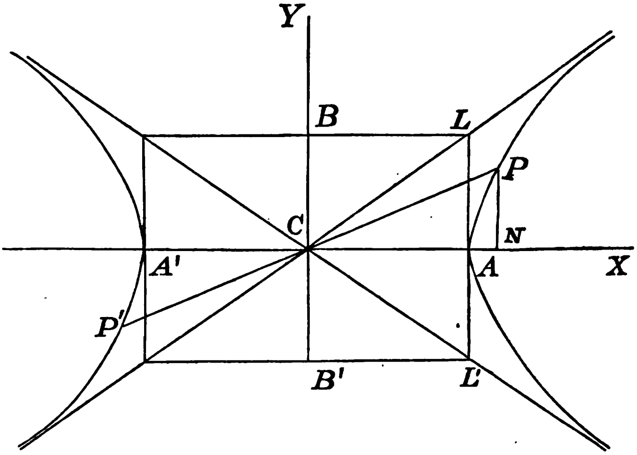
Since this equation only involves the squares of $x$ and $y$ we may prove as in Art. 95 that the origin is the centre and every chord through the origin a diameter of the curve.

**131.** The equation to the curve may be written

$$\frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2}.$$

Here for every value of $y$ there correspond two values of $x$ equal but of opposite signs, but if $x < \pm a$, $y$ is impossible; $x$ and $y$ may both increase without limit.

Hence the curve extends infinitely on both sides of the axes, and no part of it is included between the lines $x = a$, $x = -a$.



Take $C$ the centre as origin, $CX$, $CY$ the axes, make $CA$, $CA'$ each equal to $a$, $CB$, $CB'$ to $b$, then the curve passes through $A$, $A'$.

$AA'$ is called the transverse axis, $BB'$ the conjugate axis.

Let $y = x \tan\theta$ be the equation to any diameter, then, substituting in the equation to the curve,

$$x^2\left(\frac{1}{a^2} - \frac{\tan^2\theta}{b^2}\right) = 1;$$

$$\therefore \text{ if } \frac{1}{a^2} < \frac{\tan^2\theta}{b^2}, \text{ or } \tan\theta > \pm\frac{b}{a},$$

$x$ is impossible.

Through $A$ draw $LAL'$ perpendicular to $CA$, and through $B$, $B'$, $BL$, $B'L'$ parallel to $CA$; join $CL$, $CL'$ and produce these lines both ways to any distance, then

$$\tan LCA = \frac{b}{a},$$

therefore the curve lies wholly between the lines $LC$, $L'C$.

Since for every point $P$ there is another point $P'$ such that $CP' = CP$ the curve must have two branches, passing through the points $A$, $A'$ respectively, and be of the form represented in the figure.

**132.** *Asymptotes.* Definition.

Lines are said to be asymptotes to a curve, when the curve continually approaches them, but never actually reaches them, though its distance from them may be made less than any assignable distance.

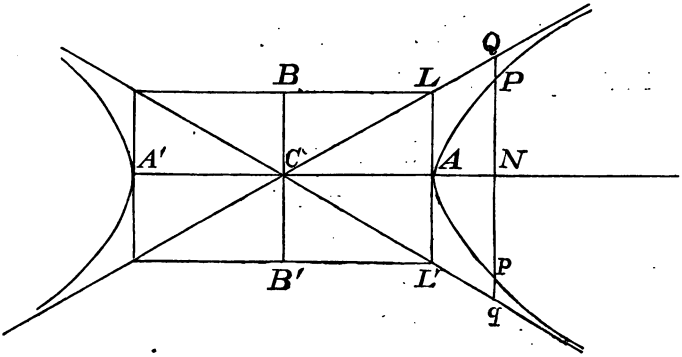
**133.** The equations to the lines $CL$, $CL'$, the diagonals of the rectangle whose sides are lines drawn through the extremities of the axes perpendicular to them, are

$$y = \pm\frac{b}{a}x,$$

$$\text{or } \frac{y^2}{b^2} = \frac{x^2}{a^2}.$$

The equation to the hyperbola is

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1.$$



Now let $P$ be a point on the hyperbola; let $CN = x_1$; let $NP$ produced both ways cut $CL$, $CL'$ produced in $Q$, $q$; let $PN = y_1$, $QN = y_2$.

Then $$\frac{y_2^2}{b^2} = \frac{x_1^2}{a^2}, \quad \frac{y_1^2}{b^2} = \frac{x_1^2}{a^2} - 1,$$

$$\therefore \frac{y_2^2 - y_1^2}{b^2} = 1, \quad \text{or } y_2^2 - y_1^2 = b^2;$$

therefore $y_1^2$ is always less than $y_2^2$, that is to say, the hyperbola never cuts the lines $CL$, $CL'$.

Also $$y_2^2 - y_1^2 = (y_2 + y_1)(y_2 - y_1) = b^2.$$

Now as $x_1$ increases without limit, $y_2$ and $y_1$ also increase without limit, and therefore $y_2 + y_1$ increases without limit.

But $$(y_2 + y_1)(y_2 - y_1) = b^2,$$

therefore $y_2 - y_1$ diminishes without limit.

Hence the distance $PQ$ diminishes without limit, hence $CL$ is an asymptote.

Similarly $CL'$ is an asymptote.

## 134. *Polar equation.* Centre pole.

In the equation to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

put $$x = r\cos\theta, \quad y = r\sin\theta,$$

then $$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1,$$

$$\text{or } r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2(e^2-1)}{e^2\cos^2\theta - 1}.$$

Here for every value of $\theta$ we have two equal and opposite values of $r$. Also the values of $r$ for $-\theta$ are the same as for $\theta$; $r$ is least when $b^2\cos^2\theta - a^2\sin^2\theta$ is greatest; that is, when $\theta = 0$; hence the least diameter is the transverse axis.

As $\theta$ increases $r$ increases, until

$$\tan^2\theta = \frac{b^2}{a^2},$$

when it becomes infinite. This gives the asymptotes.

If $\tan^2\theta > \frac{b^2}{a^2}$, $r$ is impossible, that is, no diameter beyond the asymptotes meets the curve.

**135.** When the focus is pole, we know that the equation to any conic section is

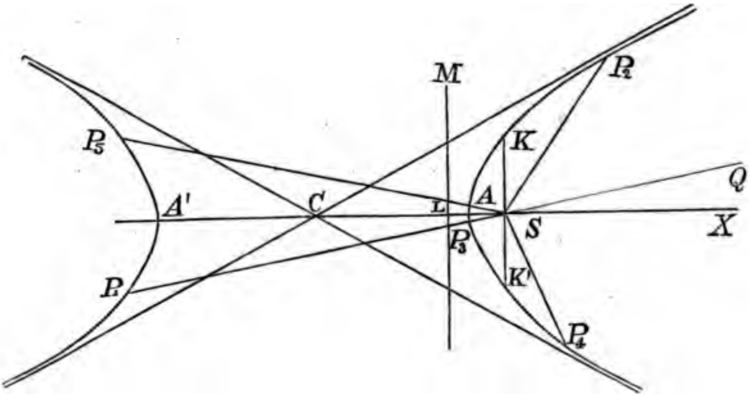
$$r = \frac{l}{1 - e\cos\theta}.$$

It is useful however to trace the hyperbola from this equation.

First let $\theta = 0$, then since $e$ is greater than 1, $1 - e\cos\theta$ is negative, and therefore $r$ negative.

Also since $\cos\theta$ diminishes when $\theta$ increases, $\frac{1}{r}$ diminishes *numerically* as $\theta$ increases and $r$ increases, and the curve is therefore convex to the pole. Hence we must begin with the further branch.

Let $QSX = \theta$,



Produce $QS$ to $P_1$ so that $SP_1 = \dfrac{l}{e\cos\theta - 1}$;

then $P_1$ is a point on the hyperbola.

Let $\theta$ increase till $\cos\theta = \dfrac{1}{e}$;

then $\cos\theta$ decreases, and

$$\frac{l}{e\cos\theta-1} \text{ increases from } -\frac{l}{(e-1)} \text{ to } \infty.$$

Hence we get the branch $AP_1$ up to the asymptote.

Next let $\theta > \cos^{-1}\frac{1}{e} < \frac{\pi}{2}$.

Then, $1-e\cos\theta$ is positive, and $r$ diminishes from $\infty$ to $l$, and we get the nearer branch $P_2B$ from the asymptote to the extremity of the latus rectum.

Next, let $\theta$ lie between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, then $\cos\theta$ is negative, and $1-e\cos\theta$ positive and greater than 1; therefore $r$ is positive, and less than $l$.

Hence we get the branch $KP_3K'$.

Next, let $\theta > \frac{3\pi}{2}$, $\cos\theta$ is positive, and $1-e\cos\theta$ negative, until $\cos\theta = \frac{1}{e}$; so we get the branch $B'P_4$ up to the other asymptote.

Lastly, let $\theta$ increase up to $2\pi$, then $\cos\theta > \frac{1}{e}$; therefore $1-e\cos\theta$ is negative, and $r$ decreases from $-\infty$ to the original value $-\frac{l}{(e-1)}$, and we get the branch $P_5A'$, from the asymptote to $A'$.

We see that whenever $e\cos\theta > 1$, and therefore $er\cos\theta$ greater than $r$, $r$ must be negative, since

$$r - er\cos\theta = l;$$

$r$ therefore must be the radius vector of some point on the other side of the directrix.

In the ellipse and parabola since $e$ is not greater that 1, $1 - e\cos\theta$ is positive, and therefore $r$ positive. Hence we see that the curve lies wholly on one side of the directrix.

### 136. *Conjugate hyperbola.*

We have seen that diameters which make an angle with the transverse axis greater than $\tan^{-1}\frac{b}{a}$, do not meet the hyperbola in real points; they however will meet the curve denoted by the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

It is evident that this curve is a hyperbola in which the axis of $y$ meets the curve in real points.

Comparing this equation with

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we see that if we interchange $x$ and $y$, $a$ and $b$, properties of the ordinates of one curve become those of the abscissæ of the other.

Hence the equation

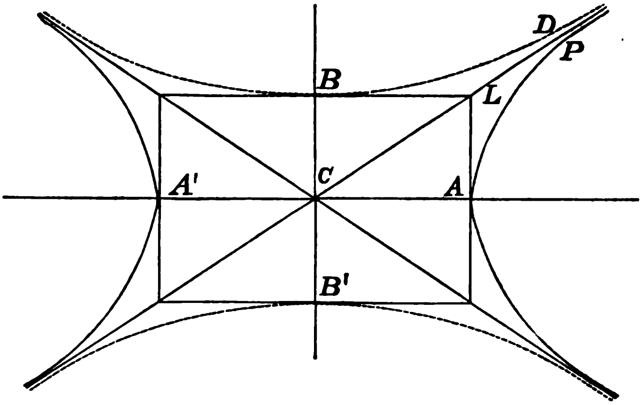
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

represents a hyperbola, having the same axes as the original hyperbola, the transverse axis of the original hyperbola being the conjugate of this, and vice versâ.

The asymptotes of this hyperbola will be

$$\frac{x}{a} = \pm\frac{y}{b}.$$

That is the two hyperbolas have the same asymptotes.



This is geometrically evident from the fact that the rectangle of which the asymptotes are diagonals, and the sides perpendiculars to the axes at their extremities is the same for both.

The polar equation to this hyperbola is evidently

$$r^2 = \frac{a^2b^2}{a^2 \sin^2\theta - b^2 \cos^2\theta}.$$

We may observe that properties of the conjugate hyperbola may be obtained from the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, by writing $-a^2$ for $a^2$. In the figure the dotted line represents the conjugate hyperbola.

**137.** In the ellipse, we put

$$x = a\cos\phi, \quad y = b\sin\phi,$$

$\phi$ being the excentric angle at the point.

To obtain the corresponding property of the hyperbola, we should have to write $-b^2$ for $b^2$, and therefore $b\sqrt{-1}$ for $b$.

Thus $\tan\phi = \frac{a}{b}\frac{y}{x}\sqrt{-1}$, which is impossible, hence there is no angle in the hyperbola which corresponds to the excentric angle in the ellipse.

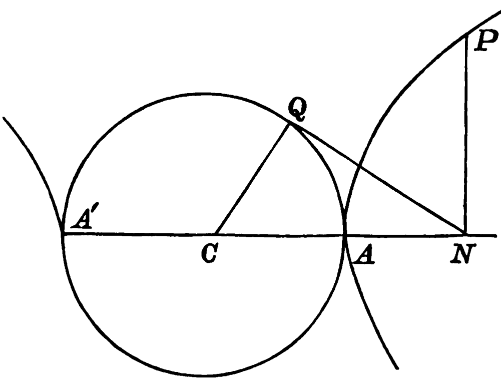
However, since $\sec^2\phi - \tan^2\phi = 1$, if in the hyperbola

$$x = a\sec\phi,\ y = b\tan\phi.$$

The geometrical angle which corresponds to $\phi$ may be thus determined.

On $AA'$ as diameter describe a circle.

Let $P$ be any point on the hyperbola, $PN$ its ordinate, draw $NQ$ touching the circle.



Then $\qquad CN = CQ \sec QCA = a\sec QCA,$

therefore if we call $QCA$, $\phi$, $x = a\sec\phi$, $y = b\tan\phi$.

Since $\qquad QN = a\tan\phi,$

$$QN : PN :: a : b.$$

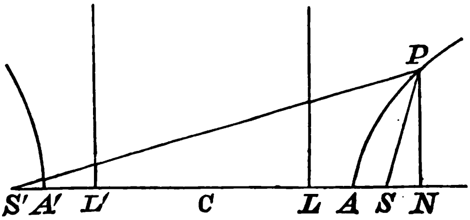
The circle described on the transverse axis as diameter is sometimes called the auxiliary circle, and the angle $\phi$ the excentric angle.

**138.** Since the axes divide the curve into two equal and similar branches, to the focus $S$ another focus $S'$ must correspond, and to the directrix $LX$ another directrix $L'X'$.

The hyperbola may be described from either focus with the corresponding directrix.

**139.** Let $P$ be any point on the curve, draw $PXX'$ perpendicular to the directrices, $PN$ perpendicular to $SS'$, and join $SP . S'P$.

Then
$$SP = e . PX = e . LN,$$
$$S'P = e . PX' = e . L'N,$$
$$\therefore S'P - SP = e . L'L.$$



Hence in the hyperbola the difference of the focal distances of any point is constant.

When $P$ coincides with $A$, $S'A - SA = AA'$,
$$\therefore S'P - SP = AA';$$

That is the difference of the focal distances of any point is equal to the transverse axis.

Hence $$CL = \frac{CA}{e} = \frac{a}{e}.$$

Now $$SA = e \, . \, AL,$$

$$\therefore SA + CA = e(AL + CL) = e \, . \, a.$$

**140.** To express the focal distances of any point in terms of the abscissa of that point.

We have $$SP = e\,LN = e(CN - CL) = e\left(x - \frac{a}{e}\right),$$

$$\therefore SP = ex - a.$$

Similarly, $$S'P = ex + a.$$

### Tangent, Normal, Diameters, &c.

**141.** In Articles 94, 96, 101, 103, 105, 106, 109, 110, 114, 115, 118, 119, 123, 124, write $-b^2$ for $b^2$ throughout, then since the proof is in no manner affected by this substitution, the propositions are true for the hyperbola.

**142.** Hence the equation to the tangent at a point $x_1\, y_1$ of a hyperbola, is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,$$

By the ordinary rules the perpendiculars from the origin on this line is

$$\frac{1}{\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right)^{\frac{1}{2}}} \text{ or } \frac{a^2b^2}{(b^4x_1^2 + a^4y_1^2)^{\frac{1}{2}}}.$$

**143.** The condition (Art. 109) that the line

$$x\cos\theta + y\sin\theta = p$$

should touch the hyperbola is

$$p = (a^2\cos^2\theta - b^2\sin^2\theta)^{\frac{1}{2}}.$$

If we transform this equation by putting $\theta = \theta' - \frac{\pi}{2}$, $\theta'$ becomes the angle which the tangent makes with the axis of $x$, and the equation becomes

$$y\cos\theta - x\sin\theta + (a^2\sin^2\theta - b^2\cos^2\theta)^{\frac{1}{2}} = 0.$$

Hence if $\tan^2\theta < \frac{b^2}{a^2}$, the line becomes impossible.

Hence every tangent makes an angle with the axis of $x$ greater than the angle made by the asymptotes.

Also if $y = 0$, $x$ is positive.

Hence every tangent cuts the axis at a point between the origin and the corresponding vertex.

**144.** Similarly, by putting $-a^2$ for $a^2$ in the corresponding equation in the ellipse, we get the tangent to the conjugate hyperbola which makes an angle $\theta'$ with the axis: its equation is therefore

$$x = y\tan\theta' - (b^2\tan^2\theta' - a^2)^{\frac{1}{2}}.$$

Now put $\tan\theta' = \cot\theta$ then $\theta$ is the angle it makes with the axis, and the equation becomes

$$y = x\tan\theta + (b^2 - a^2\tan^2\theta)^{\frac{1}{2}},$$

here

$$\tan^2\theta < \frac{b^2}{a^2}.$$

**145.** Again, in Art. 110, write $-b^2$ for $b^2$.

Then the equation to the normal at the point $(x_1 y_1)$ is

$$y - y_1 + \frac{a^2 y_1}{b^2 x_1}(x - x_1) = 0,$$

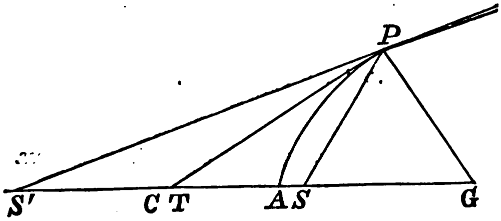
or

$$a^2 y_1 x + b^2 x_1 y = (a^2 + b^2)\, x_1 y_1.$$

**146.** Let $PG$ be the normal at $P$, $PT$ the tangent, put $y=0$ in the above equation,

$$\therefore x = x_1\left(1+\frac{b^2}{a^2}\right) = e^2x_1,$$

as in the ellipse;



$$\therefore S'G = ae + e^2x_1,\ SG = e^2x_1 - ae,$$

$$\therefore SG : S'G :: SP : S'P,$$

and $PG$ bisects the angle between $SP$ and $S'P$ produced.

Hence the tangent $PT$ bisects the angle between the focal distances.

**147.** As in the ellipse, changing $b^2$ into $-b^2$, the diameter conjugate to

$$\frac{y}{y_1} = \frac{x}{x_1},\ \text{is}\ \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 0.$$

This diameter does not meet the hyperbola at all, but it meets the conjugate hyperbola in points $\pm\frac{a}{b}y_1$, $\pm\frac{b}{a}x_1$.

Now the tangent to the hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

at the point $\frac{a}{b}y_1$, $\frac{b}{a}x_1$ is

$$\frac{yx_1 - xy_1}{ab} = 1,$$

and is $\therefore$ parallel to the diameter $\frac{y}{y_1} = \frac{x}{x_1}$.

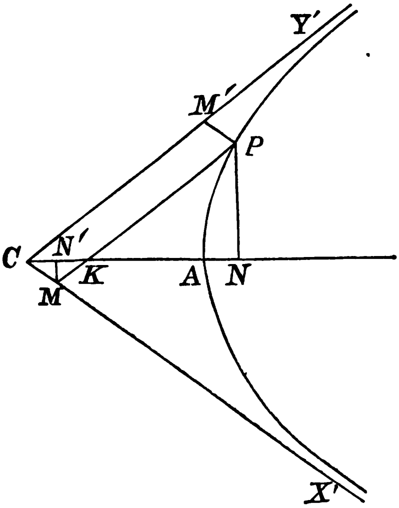
Hence if $CP$ is conjugate to $CD$, $CD$ is conjugate to $CP$.

**148.** Let $\theta$, $\theta'$, be the angles made by two conjugate diameters with the axis of $x$.

Then $$\tan\theta = \frac{y_1}{x_1},\ \tan\theta' = \frac{b^2}{a^2}\cdot\frac{x_1}{y_1},$$

$$\therefore \tan\theta\tan\theta' = \frac{b^2}{a^2}.$$

**149.** *Equation referred to the asymptotes as axes.*



Let $P$ be any point on the hyperbola, $x, y$ its coordinates referred to the old, $x', y'$ to the new axes.

Let $$Y'CA = X'CA = \alpha,$$

then $$\tan\alpha = \frac{b}{a}.$$

Now since $x' = CM = KM,\ y' = PM,$

$$\therefore\ x = CN = CN' + N'N = (CM + PM)\cos\alpha = (x' + y')\cos\alpha,$$

$$y = PN = PK\sin\alpha = (y' - x')\sin\alpha.$$

But
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$$\therefore\ \frac{(x' + y')^2}{a^2}\cos^2\alpha - \frac{(x' - y')^2}{b^2}\sin^2\alpha = 1.$$

Also
$$\cos^2\alpha = \frac{a^2}{a^2 + b^2},\ \sin^2\alpha = \frac{b^2}{a^2 + b^2},$$

$$\therefore\ (x' + y')^2 - (x' - y')^2 = a^2 + b^2,$$

or
$$4x'y' = a^2 + b^2,$$

the equation required.

**150.** *To interpret this equation geometrically.*

Draw $PM'$ parallel to $CX'$, then the area of the parallelogram

$$CMPM' = x'y \sin\omega = \frac{a^2 + b^2}{4}\sin\omega = \frac{ab}{2},$$

$$\text{since } \sin\omega = 2\sin\alpha\cos\alpha = \frac{2ab}{a^2 + b^2}.$$

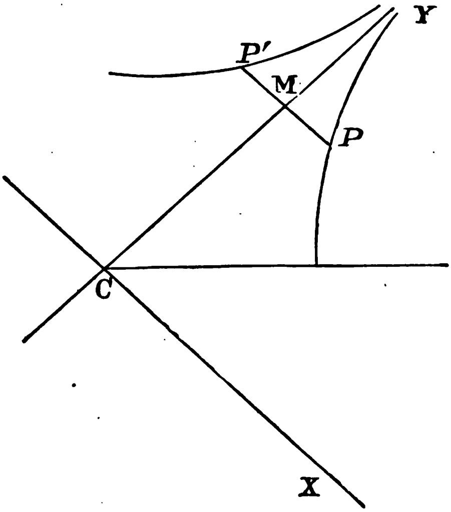
Hence if from any point on a hyperbola, straight lines be drawn parallel to the asymptotes, the area of the parallelogram thus formed is invariable.

**151.** The equation to the conjugate hyperbola is evidently

$$xy = -\frac{a^2 + b^2}{4},$$

since either $x$ or $y$ is necessarily negative, and the axes are the same as those of the original hyperbola.

**152.** Let $P$ be a point on the hyperbola; draw $PMP'$ parallel to $CX$, cutting the conjugate hyperbola in $P'$,



then $$PM = \frac{a^2+b^2}{4CM} = P'M;$$

therefore $PP'$ is bisected in $M$.

**153.** *Equation to the tangent, the asymptotes being axes.*

Let $(x_1\,y_1)$, $(x_2\,y_2)$ be two points on the hyperbola,

then $$y_1 = \frac{a^2+b^2}{4x_1}; \quad y_2 = \frac{a^2+b^2}{4x_2};$$

$$\therefore\ y_2 - y_1 = \frac{a^2+b^2}{4}\left(\frac{1}{x_2} - \frac{1}{x_1}\right) = \frac{a^2+b^2}{4}\,\frac{x_1-x_2}{x_1x_2};$$

$$\therefore\ \frac{y_2-y_1}{x_2-x_1} = -\frac{a^2+b^2}{4x_1x_2};$$

∴ the equation to the chord through $(x_1 y_1)$, $(x_2 y_2)$,

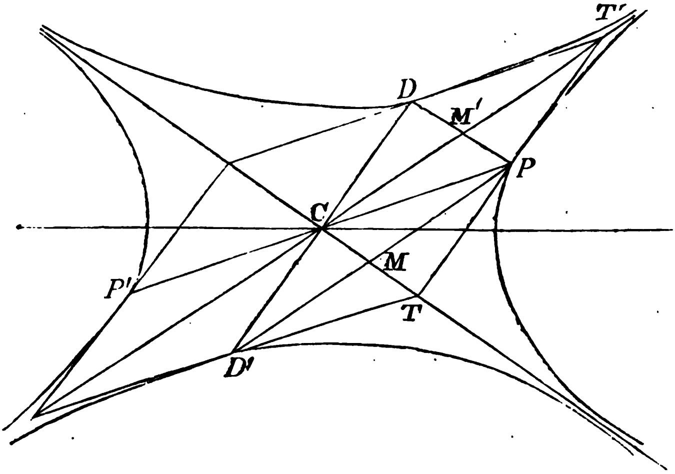
$$\text{is } y - y_1 + \frac{a^2+b^2}{4x_1 x_2}(x - x_1) = 0.$$

Now let $x_1 = x_2$ and write $\dfrac{a^2+b^2}{4y_1}$ for $x_2$.

Then the equation to the tangent becomes

$$y - y_1 + \frac{y_1}{x_1}(x - x_1) = 0,$$

or $$\frac{x}{x_1} + \frac{y}{y_1} = 2.$$



**154.** Let $P$ be the point $(x_1 y_1)$.

Let $TPT'$ be the tangent at $P$; then, putting $y_1$, $x_1$ successively equal to zero in the above equation,

$$CT = 2CM = 2x_1,$$

$$CT' = 2CM' = 2y_1;$$

therefore $T'P = TP'$, or the part of the tangent intercepted between the asymptotes is bisected at the point of contact.

**155.** The equation to the diameter conjugate to $CP$ is

$$\frac{x}{x_1} + \frac{y}{y_1} = 0,$$

since it is parallel to the tangent at $P$.

Now this cuts the conjugate hyperbola

$$4xy + a^2 + b^2 = 0,$$

in points $\pm x_1 \mp y_1$.

Hence, if $DCD'$ be this conjugate diameter, $PD$, $PD'$ are bisected by the asymptotes.

**156.** The tangent to

$$4xy + a^2 + b^2 = 0,$$

at the point $-x_1 y_1$ is

$$\frac{x}{-x_1} + \frac{y}{y_1} = 2.$$

Tangents at the extremities of conjugate diameters therefore meet on the asymptotes, which are therefore the diagonals of the parallelograms so formed.

Hence the asymptotes may be considered themselves as conjugate diameters, since each bisects chords parallel to the other.

**157.** *Rectangular Hyperbola.*

In the equation to the hyperbola let $b = a$.

Then the equation becomes

$$x^2 - y^2 = a^2.$$

In this hyperbola the asymptotes are evidently at right angles, hence it is called the rectangular hyperbola.

### EXAMPLE XII.

1. If $e_1$, $e_2$, be the excentricities of a hyperbola and its conjugate respectively,

$$\frac{1}{e_1^{\,2}} + \frac{1}{e_2^{\,2}} = 1.$$

2. The equation to the directrix is $ex = a$.

3. The circle described on the transverse axis as diameter cuts the directrices at the points where they intersect as the asymptotes.

4. The perpendiculars from the foci on the asymptotes intersect them in the directrices.

5. The lengths of these perpendiculars are equal to the conjugate semi-axis.

6. $PP'$ is a double ordinate of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$A_1 A_1'$ are the extremities of the axis major, if $AP$, $A'P'$ meet in $Q$, $Q$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

7. A circle passes through a fixed point, and cuts a given line at a given angle, shew that the locus of its centre is a hyperbola.

8. The locus of the centre of a circle, which touches two circles externally, is a hyperbola.

9. The locus of the centre of a circle, which touches one circle externally, and another internally, is a hyperbola.

10. If $S$, $H$, be the foci, $P$ any point on the hyperbola, the locus of the centre of the circle inscribed in $SPH$ is the tangent at the vertex.

11. If the circle touch $SP$, $HP$ produced, and the axis of $x$, the locus of the centre is another hyperbola.

12. The straight line $Ax + By + C = 0$ will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\text{if } A^2a^2 - B^2b^2 = C^2.$$

13. An ellipse and hyperbola have the same foci and conjugate axis, if $a$, $b$, be the semi-axes of the ellipse, $a'$, $b$ of the hyperbola, $a'^2 = a^2 - 2b^2$.

14. In q. 13, if $e_1$, $e_2$, be the excentricities of the ellipse and hyperbola respectively, $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 2$.

15. In q. 13, if $\phi_1$, $\phi_2$, be the excentric angles with reference to the ellipse and hyperbola respectively at the point of intersection,

$$\sin \phi_1 = \tan \phi_2 = \left(\frac{a^2 - a'^2}{a^2 + a'^2}\right)^{\frac{1}{2}}.$$

16. Prove that these curves cut one another at right angles.

17. The equation to a hyperbola which has the axes as asymptotes and passes through the point $(h, k)$ is $xy = hk$.

18. The straight line

$$Ax + By + C = 0$$

will touch the hyperbola $xy = c^2$, if $C^2 = 4ABc^2$.

19. If two tangents be drawn to a hyperbola, and the points in which they intersect the asymptotes be joined, the joining lines will be parallel to one another.

20. The equation to the diameter conjugate to

$$\frac{x}{h} + \frac{y}{k} = 0 \text{ is } \frac{x}{h} = \frac{y}{k},$$

the hyperbola being referred to its asymptotes.

21. If the abscissæ of any number of points in a hyperbola, referred to its asymptotes, be in A. P. the ordinates will be in H. P.

22. If on any chord as diagonal be constructed a parallelogram, the sides of which are parallel to the asymptotes, the other diagonal will pass through the centre.

23. If two hyperbolas have common asymptotes, any chord of the one touching the other will be bisected at the point of contact.

24. Tangents are drawn to a hyperbola, and the portions intercepted by the asymptotes are divided in a constant ratio; prove that the locus of the points of section is a hyperbola.

25. The coordinates of the vertex of the hyperbola $xy = c^2$, are $c$, $c$, and of the focus $c \sec \alpha$, $c \sec \alpha$, where $2\alpha$ is the angle between the asymptotes.

26. If $2\alpha$ be the angle between the asymptotes of the hyperbola $xy = c^2$, the excentricity is $\sec \alpha$.

27. The equation to the directrix of the same hyperbola is

$$x + y = 2c \cos \alpha.$$

28. The equation to the normal at the point $x_1$, $y_1$, is

$$x_1 (x + y \cos \omega - x_1) = y_1 (y + x \cos \omega - y_1).$$

29. If $A$, $S$, be the vertex and focus of a hyperbola, and the tangent at $A$, and the directrix intersect an asymptote in $E$, $R$, respectively, then $SE$ is parallel to $AR$.

30. $P$ is the point $(x_1 y_1)$ on the hyperbola $xy = c^2$, if $CP$ cut the ellipse

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1,$$

in $Q$, then the tangent to the ellipse at $Q$ is parallel to that to the hyperbola at $P$.

N.B. A very large number of the examples on the ellipse apply equally, or with the change of sign of $b^2$, to the hyperbola; those do not apply in which the first power of $b$ occurs.

## EXAMPLE XIII.

### *Rectangular Hyperbola.*

1. A diameter is equal to its conjugate.

2. The excentricity is $\sqrt{2}$.

3. If the perpendicular from the focus meet an asymptote in $R$, $SR = AC$.

4. The distance of any point from the centre is a geometric mean between its focal distances.

5. If $PN$ be the ordinate at $P$, and $NQ$ touch the auxiliary circle at $Q$, then $PN = QN$.

6. If $PG$, the normal at $P$, cut the axis in $G$, then $PG = CP$.

7. If the tangent at $P$ cut the asymptotes in $T$, $t$, respectively, then $Tt = 2CP$.

8. If $TG$, $tG$, be joined, the angle $TGt$ is a right angle.

9. The equation referred to the asymptotes is $2xy = a^2$, the coordinates being rectangular.

10. If from any point on the curve straight lines be drawn to the extremities of any diameter, these make equal angles with the asymptotes.

11. The equation referred to polar coordinates, the centre being pole, is $r^2 = a^2 \sec 2\theta$.

12. The equation to the normal at $(x_1, y_1)$ when the asymptotes are axes is the same as that to the tangent at $(x_1\, y_1)$ when the axes of the curve are axes of coordinates.

13. If the product of the tangents of the inclination to the axis of $x$ of a pair of tangents to the rectangular hyperbola $xy = c^2$ be $k$, they will intersect on the diameter $y = kx$.

14. If a right-angled triangle be inscribed in a rectangular hyperbola, the perpendicular from the right angle on the hypothenuse is a tangent to the curve.

15. If the axes be inclined at an angle $\theta$ to the axes of the curve the equation to the rectangular hyperbola is

$$(x^2 - y^2) \cos 2\theta - 2xy \sin 2\theta = a^2.$$

# CHAPTER IX.

## GENERAL PROPERTIES OF CONIC SECTIONS.

**158.** WE have already (Art. 72) seen that the polar equation to a conic section is

$$\frac{l}{r} = 1 - e\cos\theta,$$

we will now find the equation to the tangent at the point $(r_1, \theta_1)$.

First, let us find the equation to the secant passing through the two points $(r_1, \theta_1)$, $(r_2, \theta_2)$.

It is evidently

$$r\sin\theta - r_1\sin\theta_1 = \frac{r_2\sin\theta_2 - r_1\sin\theta_1}{r_2\cos\theta_2 - r_1\cos\theta_1}(r\cos\theta - r_1\cos\theta_1).$$

Now $\qquad r_1(1 - e\cos\theta_1) = r_2(1 - e\cos\theta_2) = l;$

$$\therefore\ r_2\sin\theta_2 - r_1\sin\theta_1$$

$$= \frac{r_1}{1 - e\cos\theta_2}\{(1 - e\cos\theta_1)\sin\theta_2 - (1 - e\cos\theta_2)\sin\theta_1\}$$

$$= \frac{r_1}{1 - e\cos\theta_2}\{\sin\theta_2 - \sin\theta_1 - e\sin(\theta_2 - \theta_1)\}$$

$$= \frac{2r_1\sin\frac{\theta_2 - \theta_1}{2}}{1 - e\cos\theta_2}\left(\cos\frac{\theta_2 + \theta_1}{2} - e\cos\frac{\theta_2 - \theta_1}{2}\right).$$

Similarly, $\quad r_2 \cos\theta_2 - r_1 \cos\theta_1$

$$= \frac{r_1}{1 - e\cos\theta_2}\{(1 - e\cos\theta_1)\cos\theta_2 - (1 - e\cos\theta_2)\cos\theta_1\}$$

$$= \frac{r_1}{1 - e\cos\theta_2}(\cos\theta_2 - \cos\theta_1) = -\frac{2r_1 \sin\frac{\theta_2 - \theta_1}{2}\sin\frac{\theta_2 + \theta_1}{2}}{1 - e\cos\theta_2}.$$

Hence the equation to the secant becomes

$$r\sin\theta - r_1\sin\theta_1 = \frac{e\cos\frac{\theta_2 - \theta_1}{2} - \cos\frac{\theta_2 + \theta_1}{2}}{\sin\frac{\theta_2 + \theta_1}{2}}(r\cos\theta - r_1\cos\theta_1).$$

Now, when the secant becomes the tangent, $\theta_2 = \theta_1$, therefore putting $\theta_2 = \theta_1$ in this equation, multiplying by $\sin\theta_1$, and simplifying, we obtain

$$r\cos(\theta - \theta_1) - r_1 = e(r\cos\theta - r_1\cos\theta_1),$$

the equation required.

This equation may be further simplified by dividing by $r_1 r_2$, and multiplying by $l$. Hence we obtain

$$\frac{l}{r}(1 - e\cos\theta_1) = \frac{l}{r_1}\{\cos(\theta - \theta_1) - e\cos\theta\},$$

but $\quad \dfrac{l}{r_1} = 1 - e\cos\theta_1,$

$$\therefore\ \frac{l}{r} = \cos(\theta - \theta_1) - e\cos\theta;$$

this form only involves the angular coordinate of the point of contact.

**159.** *General equation to a conic section referred to any axes.*

Let the angle between the axes be $\omega$, the coordinates of the focus $(h, k)$, and the equation to the directrix

$$x\cos\alpha + y\cos\beta = p.$$

Let the point $(x, y)$ be on the conic of which the excentricity is $e$.

Then the distance of $(x, y)$ from $(h, k)$ is

$$\{(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega\}^{\frac{1}{2}},$$

and its distance from

$$x\cos\alpha + y\cos\beta = p, \text{ is}$$

$$\pm(x\cos\alpha + y\cos\beta - p),$$

$$\therefore\ (x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega$$
$$= e^2(x\cos\alpha + y\cos\beta - p)^2,$$

the equation required.

Expanding and rearranging in terms of $x^2$, $xy$, &c. we obtain

$$x^2(1 - e^2\cos^2\alpha) + 2xy(\cos\omega - e^2\cos\alpha\cos\beta) + y^2(1 - e^2\cos^2\beta)$$
$$- 2x(h + k\cos\omega - p\cos\alpha) - 2y(k + h\cos\omega - p\cos\beta)$$
$$+ h^2 + k^2 + 2hk\cos\omega - e^2p^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the coefficients of the various terms $x^2$, $xy$, &c. can be made to vary amongst themselves as much as we please by altering the values of $\alpha$, $\beta$, $e$, $h$, $k$ or $p$, and since an equation is unaltered by multiplying it by any constant quantity, this equation may be made identical with the general equation of the second degree,

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ \ldots\ldots\ldots (2).$$

Hence we see that any conic section may be represented by the general equation of the second degree.

For the proof that the general equation always represents a conic section, for its reduction to the simple forms which we have already found, and for a full discussion of its geometrical meaning, the student is referred to Salmon's *Conic Sections*.

**160.** *To find the nature of the conic denoted by the general equation.*

Let equations (1) and (2), in the preceding Article, refer to the same curve, let $\lambda$ be the constant multiplier which makes them identical; then

$$A = \lambda(1 - e^2\cos^2\alpha), \quad C = \lambda(1 - e^2\cos^2\beta),$$

$$B = \lambda\{\cos(\alpha+\beta) - e^2\cos\alpha\cos\beta\}, \text{ since } \alpha+\beta = \omega;$$

$$\therefore AC - B^2 = \lambda^2[(1 - e^2\cos^2\alpha)(1 - e^2\cos^2\beta)$$
$$- \{\cos(\alpha+\beta) - e^2\cos\alpha\cos\beta\}^2]$$
$$= \lambda^2[1 - \cos^2(\alpha+\beta) - e^2\{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)\}].$$

Now $\quad \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$

$$= 1 + \frac{\cos 2\alpha + \cos 2\beta}{2} - \cos(\alpha+\beta)\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) - \cos(\alpha+\beta)\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$= 1 - \cos^2(\alpha+\beta) = \sin^2\omega, \text{ since } \alpha+\beta = \omega;$$

$$\therefore AC - B^2 = \lambda^2(1 - e^2)\sin^2\omega.$$

Now, the curve will be an ellipse, parabola, or hyperbola, as $e <, = 1$, or $> 1$, that is, as $1 - e^2 >, =,$ or $< 0$, hence the curve represented by the general equation will be an ellipse, parabola, or hyperbola, according as

$$AC - B^2 \text{ is } >, =, \text{ or } < \text{ zero.}$$

**161.** *To find the equation to a conic referred to two tangents as axes.*

Let the equation be

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \;\ldots\ldots\; (1).$$

Let $a$, $b$, be the distances of the points of contact from the origin, respectively.

In equation (1) put $y = 0$; then, since the axis of $x$ touches the curve, the equation must reduce to a quadratic, both the roots of which are equal to $a$; therefore the equation

$$Ax^2 + 2Dx + E = 0$$

must have two roots each equal to $a$;

$$\therefore \frac{D}{A} = -a, \quad \frac{E}{A} = a^2, \quad \text{and therefore } \frac{F}{D} = -a.$$

Similarly,

$$\frac{E}{C} = \frac{F}{E} = -b, \quad \frac{F}{C} = b^2;$$

therefore dividing by $F$, and substituting the values of the coefficients thus determined,

$$\frac{x^2}{a^2} + \frac{2B}{F}xy + \frac{y^2}{b^2} - \frac{2x}{a} - \frac{2y}{b} + 1 = 0,$$

$$\text{or } \left(\frac{x-a}{a}\right)^2 + \left(\frac{y-b}{b}\right)^2 = 1 - \frac{2B}{F}xy = 1 + 2\lambda\frac{xy}{ab},$$

where
$$\frac{\lambda}{ab} = \frac{-B}{F}.$$

Here the curve is an ellipse, parabola or hyperbola, as $1 - \lambda^2 >, =, \text{ or } < 0$.

The above equation may also be written

$$\frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} - 2\left(\frac{x}{a} + \frac{y}{b}\right) + 1 = 2(1+\lambda)\frac{xy}{ab};$$

$$\text{or } \left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = \frac{2(1+\lambda)xy}{ab}.$$

Putting $1 + \lambda = \frac{\mu^2}{2}$, we obtain the equation in the form

$$\frac{x}{a} + \frac{y}{b} - 1 = \mu\left(\frac{xy}{ab}\right)^{\frac{1}{2}};$$

Here, since $\lambda^2 =$ or $> 1$ in the parabola or hyperbola, and $2(1+\lambda) = \mu^2$, $\lambda$ must be positive, or the equation will either represent a straight line or become impossible.

**162.** In the case of the parabola this equation admits of a still further simplification.

Here $\lambda = 1$; hence the equation becomes

$$\frac{x}{a} + 2\left(\frac{xy}{ab}\right)^{\frac{1}{2}} + \frac{y}{b} = 1,$$

where the root may have either sign.

Extracting the square root of both sides, we obtain

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1,$$

where the roots may have either sign.

There is no necessity for expressing the ambiguities in this equation, since they vanish when the equation becomes rational.

**163.** *Any straight line through a pole of a conic section is harmonically divided by the curve and the polar.*

Take the pole as origin, and let the axes be the tangents. Then the equation to the curve may be written

$$\frac{x^2}{a^2} + 2\lambda\frac{xy}{ab} + \frac{y^2}{b^2} - 2\left(\frac{x}{a} + \frac{y}{b}\right) + 1 = 0 \text{ ............ (1)},$$

and the equation to the polar is

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ ........................ (2)},$$

since it passes through the points of contact of the tangents.

Let the equation to any straight line through the pole be

$$\frac{x}{m} = \frac{y}{n} = r,$$

where $r$ is the length from the origin.

Substitute $mr$, $nr$, for $x$ and $y$ respectively in equation (1), and divide by $r^2$;

$$\therefore \frac{m^2}{a^2} + 2\lambda \frac{mn}{ab} + \frac{n^2}{b^2} - 2\left(\frac{m}{a} + \frac{n}{b}\right)\frac{1}{r} + \frac{1}{r^2} = 0.$$

Now, if $\frac{1}{r_1}$, $\frac{1}{r_2}$ be the roots of this quadratic in $\frac{1}{r}$, that is, $r_1$, $r_2$ the values of $r$ at the points where the line $\frac{x}{m} = \frac{y}{n} = r$ cuts the conic,

$$\frac{1}{r_1} + \frac{1}{r_2} = 2\left(\frac{m}{a} + \frac{n}{b}\right):$$

but if $r_3$ be the value of $r$ where this line cuts the polar (2),

$$\frac{mr_3}{a} + \frac{nr_3}{b} = 1, \text{ or } \frac{m}{a} + \frac{n}{b} = \frac{1}{r_3};$$

$$\therefore \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{r_3},$$

that is, $r_3$ is a harmonic mean between $r_1$ and $r_2$.

## EXAMPLE XIV.

1. The tangents at the extremities of the transverse axis of any conic intersect any other tangent in two points; prove that the straight lines joining these points with the focus are at right angles to each other.

2. The intersection of any tangent with the straight line drawn at right angles to the radius vector lies on the directrix.

3. If the angle between the focal distances of two points on the curve

$$\frac{l}{r} = 1 - e \cos \theta$$

be $\alpha$, the tangents at those points will intersect on the curve

$$\frac{l}{r} = \cos \alpha - e \cos \theta.$$

4. The equation to the chord whose angular points are $\alpha + \beta$, $\alpha - \beta$, is

$$\frac{l}{r} = \sec \beta \cos (\theta - \alpha) - e \cos \theta.$$

5. If two conic sections touch one another at two points they cannot intersect at any other point.

6. The coordinates of the centre of the conic

$$\left(\frac{x-a}{a}\right) + \left(\frac{y-b}{b}\right)^2 = 1 + 2\lambda \frac{xy}{ab}$$

are $\dfrac{a}{1-\lambda}$, $\dfrac{b}{1-\lambda}$.

7. The equation to this conic transferred to parallel axes through the centre is

$$\frac{x^2}{a^2} - \frac{2\lambda xy}{ab} + \frac{y^2}{b^2} = \frac{1+\lambda}{1-\lambda}.$$

8. The length of the diameter through the origin is

$$\frac{\{2(1+\lambda)(a^2 + 2ab\cos\omega + b^2)\}^{\frac{1}{2}}}{1-\lambda}.$$

9. The equation

$$\left(\frac{x-a}{a}\right)^2 + \left(\frac{y-b}{b}\right)^2 = 1 + 2\frac{xy}{ab}\operatorname{cosec}^2\omega$$

represents a rectangular hyperbola.

10. The diameters of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

are parallel to the line

$$\frac{x}{a} = \frac{y}{b}.$$

11. The straight line

$$\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$$

is a tangent to

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1;$$

at the point $\left(\frac{a}{4}, \frac{b}{4}\right)$.

12. The coordinates of the focus of the same parabola are

$$\frac{ab^2}{a^2 + 2ab\cos\omega + b^2}, \quad \frac{a^2b}{a^2 + 2ab\cos\omega + b^2},$$

of the vertex

$$\frac{ab^2(a\cos\omega + b)^2}{(a^2 + 2ab\cos\omega + b^2)^2}, \quad \frac{ab(a + b\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2},$$

and the length of the latus rectum is

$$\frac{4a^2b^2\sin^2\omega}{(a^2 + 2ab\cos\omega + b^2)^{\frac{3}{2}}}.$$

## MISCELLANEOUS PROBLEMS AND EXAMPLES.

1. The points $(a, -b)$, $(3a, 3b)$, $(3a, b)$, $(5a, 5b)$, are the angular points of a parallelogram.

2. The cosine of the angle between $lx + my = 1$, and $l'x + m'y = 1$, is

$$\frac{ll' + mm'}{(l^2 + m^2)^{\frac{1}{2}} + (l'^2 + m'^2)^{\frac{1}{2}}}.$$

3. $O$ is a fixed point; $Q$, $R$, move along fixed straight lines so that $QOR$ is constant; $P$ is a point in $QR$ such that $POQ$ is constant; prove that the locus of $P$ is a straight line.

4. The straight lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, are perpendicular to one another, whatever the inclination of the axes, if

$$AA' + BB' = AB' + A'B = 0.$$

5. They will be equally inclined to the axis of $x$ in opposite directions, if

$$\frac{B}{A} + \frac{B'}{A'} = 2 \cos \omega.$$

6. They will be equally inclined to the axes of $x$, perpendicular to each other, and pass through the origin, if they are represented by the equation

$$x^2 + 2xy \cos \omega + y^2 \cos 2\omega = 0.$$

7. If the angle between $\frac{x}{l} = \frac{y}{m}$ and $\frac{x}{l'} = \frac{y}{m'}$ be $\frac{\pi}{2} - \omega$,

$$(ll' + mm') \cos \omega + (lm' + l'm) \cos^2 \omega \pm (lm' - l'm) \sin^2 \omega = 0.$$

8. The straight lines bisecting the angle between those represented by the equation

$$ax^2 + 2bxy + cy^2 = 0,$$

are represented by the equation

$$b(x^2 - y^2) = (a - c)\, xy.$$

9. The distance of the point $(x_1, y_1)$ from each of two straight lines through the origin is $\delta$; prove that the two straight lines may be represented by the equation

$$(x_1 y - x y_1)^2 = (x^2 + y^2)\, \delta^2.$$

10. Shew that the equation

$$\frac{x^2}{a^2} + \frac{b^2}{y^2} + \frac{a^2 y^2}{b^2 x^2} = \frac{y^2}{b^2} + \frac{a^2}{x^2} + \frac{b^2 x^2}{a^2 y^2}$$

represents the sides and angles of a parallelogram.

11. The polar equations to four straight lines are

$$\frac{c}{r} = A \cos(\theta - \alpha) + B \sin(\theta - \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\frac{c}{r} = A' \cos(\theta - \alpha) + B' \cos(\theta - \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\frac{c}{r} = A \sin(\theta - \alpha) - B \cos(\theta - \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\frac{c}{r} = A' \sin(\theta - \alpha) + B' \cos(\theta - \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

shew that (3) and (4) are perpendicular to (1), (2) respectively.

12. From a given point $C$ a straight line $CQ$ is drawn to a given line, and a point $P$ taken in it such that $CP.CQ = \text{constant}$; the locus of $P$ is a circle.

13. Chords of a given circle are drawn to touch another given circle; the centre of the first circle being pole, the radii joining the centres initial line, and equal in length to $c$, and the radius of the second circle being $a$, the equation to the middle points of the chords is

$$r = a + c\cos\theta.$$

14. If a triangle $ABC$ circumscribe a circle, and $p_1, p_2, p_3$ be the perpendiculars from any point on $OA$, $OB$, $OC$,

$$p_1\cos\frac{A}{2} + p_2\cos\frac{B}{2} + p_3\cos\frac{C}{2} = 0.$$

15. The locus of a point, the sum of the squares of the perpendiculars from which on the sides of a given polygon is constant, is a circle.

16. On any chord of a parabola as diameter a circle is described, cutting the parabola again in two points; if these two points be joined, the portion of the axis included between the two chords is equal to the latus rectum.

17. $PSQ$ is a focal chord of a parabola, $QNQ'$ is drawn perpendicular to the axis cutting the curve again in $Q'$; prove that $PQ'$ passes through the foot of the directrix.

18. A parabola always touches two given straight lines and has the direction of its axis fixed; the locus of the focus is a straight line.

19. If a triangle be inscribed in a parabola, and a similar one described about it, the sides of the former are four times those of the latter.

20. Two normals at right angles meet in $O$, $ON$ is drawn perpendicular to the axis, and $NQ$ measured along the axis from the vertex equal to $a$; $QO$ is a normal to the parabola.

21. $YPR$ is a straight line of which $Y$ lies on the tangent at the vertex, $P$, its middle point, on the parabola, and $R$ on the axis, prove that $YPR$ always touches another parabola.

22. The locus of points from which pairs of tangents being drawn intercept constant lengths on the tangent at the vertex, is an equal parabola, with the same axis.

23. Two equal parabolas have the same axis and vertex but are turned in opposite directions; chords of one are tangents to the other; shew that the locus of the middle points of the chords is a parabola, whose latus rectum is one-third that of either parabola.

24. A quadrilateral circumscribes a parabola: the radical axis of the circles described on its diagonals is the directrix.

25. The equation

$$(x+a)^{\frac{1}{2}} + (y+a)^{\frac{1}{2}} = (2a)^{\frac{1}{2}}$$

represents a parabola, of which the vertex is origin, the axis bisects the angle between the axes, and the latus rectum is $4\sqrt{2}a$.

26. From $P$, a point in an ellipse, straight lines are drawn to $A$, $A'$, the extremities of the axis major, and from $A$, $A'$, straight lines $AQ$, $A'Q'$ are drawn perpendicular to these lines intersecting in $R$, the locus of $R$ is the ellipse

$$\frac{x^2}{a^2} + \frac{b^2y^2}{a^4} = 1.$$

27. The locus of the extremity of the straight line formed by adding the abscissa to the ordinate of any point on the circle $x^2 + y^2 = c^2$, is an ellipse.

28. The locus of the intersection of two tangents to a circle, such that the product of the distances from the centre cut off from a given diameter is constant, is an ellipse.

29. In the axes $AB$, $AC$, are taken two points $P$, $Q$, such that $AP : PB :: AQ : QC$; the locus of the intersection of $PC$, $QB$ is an ellipse.

30. The curves

$$(x^2+y^2)^{\frac{1}{2}} = x\sin\theta - y\cos\theta,$$

and

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2},$$

intersect on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

31. A triangle $ABC$ has two angular points on the rectangular axes; the locus of the third angle is an ellipse.

32. The equation

$$\left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right)$$

represents the tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

from the point $(\alpha, \beta)$.

33. $SQ$, $HQ$, are drawn perpendicular to a pair of conjugate diameters and intersect in $Q$; the locus of $Q$ is a concentric ellipse.

34. A series of ellipses have their equal conjugate diameters of the same magnitude, one being common while the other varies in position; prove that tangents drawn from any point

in the fixed diameter produced will touch the ellipse in points situated in a circle.

35. If a parallelogram touch an ellipse at the extremities of conjugate diameters, the equation to one of which is $ly = mx$, shew that the equation

$$\pm \frac{ay}{bx} = \left(\frac{ma - lb}{ma + lb}\right)^{\pm 1}$$

determines its diagonals.

36. The locus of the vertices of an equilateral triangle about the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is $4\,(b^2x^2 + a^2y^2 - a^2b^2) = 3\,(x^2 + y^2 - a^2 - b^2)^2$.

37. A triangle is inscribed in an ellipse so that the centre of the inscribed circle is one of the foci, shew that its radius is

$$\frac{l}{1 + (1 + e^2)^{\frac{1}{2}}}.$$

38. Two triangles $ABC$, $A'B'C'$, are described about an ellipse; $B'C'$ is parallel to $BC$, $C'A'$ to $CA$, and $A'B'$ to $AB$; if any tangent cut these sides of $A'B'C'$ in $P$, $Q$, $R$, respectively, $AP$, $BQ$, $CR$ will be parallel.

39. Pairs of tangents at right angles are drawn to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

prove that the locus of the middle points of their chord of contact is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2 + b^2}.$$

40. If to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

there be drawn the four tangents

$$\frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} = \pm 1,$$

$$x \sin \phi - y \cos \phi = \pm (a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{\frac{1}{2}},$$

their points of intersection will lie on the ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = a + b,$$

and the perimeter of the parallelogram will be $4(a + b)$.

41. The length of the perpendicular let fall from the centre on a chord joining two diameters at right angles is

$$\frac{ab}{(a^2 + b^2)^{\frac{1}{2}}}.$$

42. $TP$, $TQ$ are two tangents to an ellipse at right angles: shew that $\sin^2 SPT + \sin SQT$ is constant.

43. If a circle be described through the two foci, cutting an ellipse, the angle between the circle and ellipse at either point of intersection is equal to the angle between the normal to the ellipse and the axis minor.

44. On any straight line through the origin $C$, three points $P$, $Q$, $R$, are taken such that $CP \,.\, CR = CQ^2$; if the loci of $Q$ and $R$ be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{a^2 x^2}{b^4} + \frac{y^2}{b^2} = 1, \text{ respectively,}$$

that of $P$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{a^2}{b^2}\left(\frac{x^2}{b^2} + \frac{y^2}{a^2}\right).$$

45. A straight line from the centre meets an ellipse in $P$, and the auxiliary circle in $Q$, shew that the locus of the tangents at $P$ and $Q$ is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \frac{y^2}{b^2}\left(\frac{x}{a} \pm \frac{b\,(x^2 - a^2)^{\frac{1}{2}}}{a^2}\right)^2.$$

46. From each point of the circle $x^2 + y^2 = c^2$, a straight line is drawn making an angle $\alpha$ with the radius at that point; shew that the middle points of the parts of those lines intercepted by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

lie on the curve

$$\frac{a^2\,(c^2 \sin^2 \alpha - y^2)}{b^2 x^2} + \frac{b^2\,(c^2 \cos^2 \alpha - x^2)}{a^2 y^2} = 2.$$

47. If one focus, a point, and the length of the axis minor of an ellipse be given, the locus of the other focus is a parabola.

48. $PGg$ is a normal cutting the axes in $G, g$; prove that

$$PSg = PGH.$$

49. The locus of the intersection of tangents to the ellipse whose excentric angles differ by $\beta$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \frac{\beta}{2}.$$

50. If from a point $O$ be drawn four normals, $OP$, $OQ$, $OR$, $OS$, and $p$, $q$, $r$, $s$, be taken such that the coordinates of $p$, $q$, $r$, $s$, are the intercepts on the axes of tangents at $P$, $Q$, $R$, $S$, respectively, $p$, $q$, $r$, $s$, lie in a straight line.

51. If $PCP'$, $DCD'$, two conjugate diameters, subtend angles $\alpha$, $\beta$, at either extremity of the axis minor,

$$CP^2 = \frac{a^2 \tan^2 \beta + b^2 \tan^2 \alpha}{\tan^2 \alpha + \tan^2 \beta}.$$

52. The normal at $P$ meets the curve at $Q$, $PP'$ is any chord, $QP'$ meets a line through $P$ perpendicular to $PP'$ in $R$, shew that the locus of $R$ is

$$\frac{x \cos \phi}{a} - \frac{y \sin \phi}{b} = \frac{a^2 + b^2}{a^2 - b^2},$$

where $\phi$ is the excentric angle at $P$.

53. If the four normals at $P$, $Q$, $R$, $S$ meet in a point, the sum of their excentric angles is an odd multiple of $\pi$, and the tangents parallel to $PQ$ and $RS$ meet on the equal conjugate diameters.

54. If $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be the equation to an ellipse referred to conjugate diameters, the circle

$$x^2 + 2xy \cos \omega + y^2 = r^2$$

will touch it, if

$$\left(\frac{1}{r^2} - \frac{1}{a^2}\right)\left(\frac{1}{r^2} - \frac{1}{b^2}\right) = \frac{\cos^2 \omega}{r^4}.$$

55. On a chord of an ellipse a circle is described, shew that the normals at the two other common points intersect the chord in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.$$

56. If perpendiculars be drawn from the centre of an ellipse on two tangents at right angles, the semi-diameters equal

in length to these perpendiculars respectively are conjugate, and the ordinates at their extremities meet these perpendiculars in the auxiliary circle.

57. If the tangents through a pole include with the polar a constant area, the pole and polar lie on concentric similar ellipses.

58. If through two given points, lines be drawn parallel to a pair of conjugate diameters, the locus of their intersection will be an ellipse through the points similar to the original ellipse.

59. Two straight lines are chalked on the table, and a shilling and half-crown are placed one touching each line, and touching each other, the locus of their point of contact is an ellipse.

60. Chords are drawn at right angles to each other through the vertex of an ellipse; the pole of the line joining their other points of section lies on a straight line.

61. Two conjugate diameters of an ellipse are drawn, and their extremities are joined to any point on a concentric circle of given radius; shew that the sum of the squares on these four lines is constant.

62. From any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

tangents are drawn to the ellipse

$$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1;$$

prove that if a straight line be drawn through the centre parallel to the tangent to the outer ellipse at the point, the portion of each tangent cut off will be $\frac{ab}{\lambda}$.

63. A chord of an ellipse is drawn through a fixed point, and a conic touches the ellipse at the extremities of this chord, and passes through the centre; the locus of the centre of this conic is a conic.

64. Two tangents are drawn to the parabola $y^2 = 4ax$; prove that, if the product of the cosines of their inclinations to the axis be $k$, their point of intersection will lie on the ellipse

$$k^2x^2 = (x-a)^2 + y^2.$$

65. Two tangents to one hyperbola are asymptotes to another; if that other hyperbola touch one asymptote of the former, it will also touch the other asymptote.

66. A series of hyperbolas having the same asymptotes is cut by a straight line parallel to an asymptote, and through the points of section straight lines are drawn parallel to the other asymptote, and equal to a semi-axis; the locus of their extremities is a parabola.

67. The locus of the points of intersection of two tangents to an ellipse which intercept, on any diameter, distances from the centre, the product of which is constant, is a concentric hyperbola, and two arcs of a concentric ellipse.

68. If a perpendicular be drawn from $(\alpha, \beta)$ on any tangent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the locus of the point of intersection is the curve

$$\{x(x-\alpha) + y(y-\beta)\}^2 = a^2(x-\alpha)^2 - b^2(y-\beta)^2.$$

69. A hyperbola of given excentricity has its centre on a given circle; one asymptote passes through a fixed point on the circumference: prove that the other asymptote and the

transverse axis will also pass through fixed points on the circumference of the circle.

70. The base of a triangle $ABC$ being fixed, $C$ moves on a rectangular hyperbola through $A$, $B$; if $P$, $Q$ be the points in which $AC$, $BC$ intersect the circle on $AB$ as diameter, $AQ$, $BP$ intersect on the other branch of the hyperbola.

71. If the straight lines which bisect the angles between pairs of tangents to an ellipse be parallel to a fixed line, the locus of the points of intersection is a rectangular hyperbola.

72. $S$, $H$, are the foci of an ellipse, and the extremities of a diameter of a rectangular hyperbola; shew that the tangent and normal to the ellipse, where the hyperbola meets it, are parallel to the asymptotes.

73. If a tangent be drawn to one of two confocal conics perpendicular to a tangent to the other, the locus of their intersection is a circle.

74. The equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + 1 = 0$$

will represent a point if

$$AC - B^2 = D^2C - 2BED + AE^2 = 0.$$

75. If $\xi$, $\eta$, are the coordinates of a point in the preceding question,

$$D\xi + E\eta + 1 = 0.$$

76. If the extremities of the base of a triangle move along fixed lines, the locus of the vertex is a conic section.

77. If the vertical angle be the supplement of that between the straight lines, this conic section becomes a straight line.

78. The polar of $(r_1, \theta_1)$ with respect to the conic

$$\frac{l}{r} = 1 + e \cos \theta,$$

$$\text{is} \quad \left(\frac{l}{r} - e \cos \theta\right) = \cos (\theta - \theta_1).$$

79. A parabola is drawn to touch three given straight lines; prove that the chords of contact pass each through a fixed point.

80. If a tangent to

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

meet the axes in $P$, $Q$, and perpendiculars be drawn from $P$, $Q$ to the opposite axes, they will intersect on the line

$$\frac{x + y \cos \omega}{a} + \frac{y + x \cos \omega}{b} = \cos \omega.$$

81. The directrix of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

is

$$(a + b \cos \omega)\, x + (b + a \cos \omega)\, y = a \cos \omega.$$

82. A parabola touches three sides of a triangle $ABC$: prove that if it touches $AB$, $AC$, in $P$, $Q$, respectively, $PQ$ will touch any conic section of which $AB$, $AC$ are conjugate diameters.

83. The locus of the centre of a conic section which touches three given straight lines, and passes through a given point, is another conic.

84. If the straight line

$$\frac{x}{\alpha}+\frac{y}{\beta}=1$$

touch the conic

$$\frac{x^2}{a^2}+\frac{2xy}{mn}+\frac{y^2}{b^2}+\frac{2x}{a}+\frac{2y}{b}+1=0,$$

then $$\alpha\beta\left(\frac{1}{ab}+\frac{1}{mn}\right)+2\left(\frac{\alpha}{a}+\frac{\beta}{b}+1\right)=0.$$

85. If a conic section touch four straight lines, its centre lies on a straight line.

86. The equation to any conic section referred to the vertex as origin and the axis and tangent as axes, is

$$y^2=mx+nx^2.$$

87. *CP* is a semidiameter meeting the directrix in *Q*, *S* is a focus; *SQ* is perpendicular to the tangent at *P*.

88. In any conic, if $r$, $r'$ be focal distances at right angles, and $2l$ the latus rectum,

$$\left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}.$$

89. The length of the diameter through the origin of the conic

$$\frac{x}{a}+\frac{y}{b}-1=\frac{2(xy)^{\frac{1}{2}}}{c} \text{ is } \frac{c\{ab(a^2+b^2)\}^{\frac{1}{2}}}{c^2-ab}.$$

90. The locus of the intersection of the radius vector with the perpendicular from the foot of the directrix on the tangent is a circle.

91. In the general equation, if the axes are rectangular, $\frac{B^2 - AC}{(A + C)^2}$ is constant.

92. Prove that the locus of the centre of a conic through four given points is a conic, and that when the four points are on a circle, the locus becomes a rectangular hyperbola.

93. Pairs of normals are drawn at the extremities of chords of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation to any chord being

$$\frac{x \cos \alpha}{a} + \frac{y \sin \alpha}{b} = \lambda,$$

shew that the locus of their points of intersection is the hyperbola

$$2 (ax \sin \alpha + by \cos \alpha) (ax \cos \alpha + by \sin \alpha)$$
$$= (a^2 - b^2)^2 \sin 2\alpha \cos^2 2\alpha.$$

94. If $2\alpha$, $2\beta$ be the axes of

$$a^2 (x + y)^2 + 2b^2 xy = c^2,$$

then $$(\alpha^2 + \beta^2) a^2 + (\alpha^2 - \beta^2) b^2 = 2c^4.$$

95. If $$Ax^2 + 2Bxy + Cy^2 = 1$$

represent a rectangular hyperbola,

$$A + C = 2B \cos \omega.$$

96. Given a focus, tangent, and latus rectum $2l$ of a conic section, prove that the locus of the other focus is

$$l^2 \{x^2 + (c - y)^2\} = c^2 y^2,$$

where the tangent is the axis of $x$, and the focus is the point $(0, c)$.

97. A straight line makes an angle $\alpha$ with the radius vector of a conic, and meets the tangent at the extremity of that radius vector: the locus of its intersection is an ellipse, parabola, or hyperbola, as $\cos\alpha >=$ or $< e$.

98. The equation to the parabola whose focus is the origin, and directrix

$$\frac{x}{a}+\frac{y}{b}=1,$$

$$\text{is } \frac{x^2}{b^2}-\frac{2xy}{ab}+\frac{y^2}{a^2}+\frac{2x}{a}+\frac{2y}{b}=1.$$

99. All chords of a conic which subtend a right angle at a given point on the curve, intersect in the normal at that point.

100. If $I$ be this point on the normal, $PI$ varies as the diameter conjugate to $CP$.

## LIST OF FORMULÆ.

N.B. The axes are rectangular, unless the contrary is stated.

The distance between $(x_1, y_1)$ and $(x_2, y_2)$ is $\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}^{\frac{1}{2}}$.

The area of triangle whose angular points are given is

$$\pm\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

*Transformations.* Origin changed but not axes:

$$x = x' + h, \quad y = y' + k.$$

Axes changed but not origin,

$$x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta.$$

Also $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$.

$x = mr$, $y = nr$, from oblique to polar coordinates.

*Equations to the straight line.*

$Ax + By + C = 0$, general equation, any axes:

$\dfrac{x}{a} + \dfrac{y}{b} = 1$, in terms of intercepts, any axes:

$A(x - x_1) + B(y - y_1) = 0$, through point $(x_1, y_1)$:

$\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$, through points $(x_1, y_1)$, $(x_2, y_2)$:

$$x\cos\alpha + y\sin\alpha = p;$$

$p$, the perpendicular from origin, makes angle $\alpha$ with the axis of $x$,

$x\cos\alpha + y\cos\beta = p$; axes contain the angle $\alpha + \beta$:

$\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = r$, where $r$ is the distance from $(x_1, y_1)$,

$$r\cos(\theta - \alpha) = p,$$

$p$ distance from pole, making an angle $\alpha$ with the initial line.

The distance of $(x_1, y_1)$ from $Ax + By + C = 0$ is

$$\frac{Ax_1 + By_1 + C}{(A^2 + B^2)^{\frac{1}{2}}};$$

from $x \cos \alpha + y \sin \alpha = p$, $x_1 \cos \alpha + y_1 \sin \alpha - p$.

If two straight lines are parallel $\frac{A}{B} = \frac{A'}{B'}$,

if perpendicular $\frac{A}{B} + \frac{B'}{A'} = 0$.

## CIRCLE.

*Equations:* $x^2 + y^2 = c^2$, centre origin;

$(x - a)^2 + (y - b)^2 = c^2$, centre $(a, b)$;

$x^2 + y^2 + 2Ax + 2By + C = 0$, general equation.

$r = l \cos (\theta - \alpha)$, polar equation, pole on the curve.

*Equations to tangent;* at the point $(x_1, y_1)$

$$xx_1 + yy_1 = c^2;$$

to any tangent, $x \cos \theta + y \sin \theta = c$.

$hx + ky = c^2$ is the polar of $(h, k)$,

$\frac{c^2 \cos \theta}{p}$, $\frac{c^2 \sin \theta}{p}$ is the pole of $x \cos \theta + y \sin \theta = p$.

$S_1 - S_2 = 0$, radical axis of $S_1 = 0$, $S_2 = 0$.

## PARABOLA.

$y^2 = 4ax$, equation to curve.

$yy_1 = 2a (x + x_1)$ to the tangent at $(x_1\ y_1)$.

$y = x \tan \theta + a \cot \theta$ to any tangent making an angle $\theta$ with the axis.

$x + a$, focal distance of $(x, y)$.

$y^2 = 4a'x$, equation to curve referred to a diameter and its tangent.

$\frac{2a}{r} = 1 - \cos \theta$, polar equation.

### ELLIPSE.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ referred to axes.}$$

$$x^2 + y^2 = a^2, \text{ the auxiliary circle.}$$

$$x = a\cos\phi,\ y = \overset{b}{\cancel{a}}\sin\phi,\ \phi \text{ the excentric angle.}$$

$$(a + ex),\ (a - ex), \text{ focal distances of } (x, y).$$

*Equations to the tangent:*

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \text{ at point } (x_1, y_1),$$

$$\frac{x\cos\phi_1}{a} + \frac{y\sin\phi_1}{b} = 1, \text{ at point at which } \phi = \phi_1,$$

$$x\cos\theta + y\sin\theta = (a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}, \text{ general equation.}$$

The length of semidiameter is

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}},\ \frac{b}{(1 - e^2\cos^2\theta)^{\frac{1}{2}}} \text{ or } \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^{-\frac{1}{2}}.$$

Length of conjugate diameter

$$(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}}, \text{ or } \frac{(a^4y^2 + b^4x^2)^{\frac{1}{2}}}{ab}.$$

The coordinates of the extremity of the diameter conjugate to that through $(x_1, y_1)$ are $\pm\frac{a}{b}y_1$, $\mp\frac{b}{a}x_1$, and its excentric angle is $\frac{\pi}{2} + \phi$.

The equation referred to any conjugate diameters is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1:$$

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}, \text{ polar equation, centre pole:}$$

$4ab$, area of circumscribing parallelogram.

## HYPERBOLA.

Change $b^2$ into $-b^2$ in the ellipse;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

$x = a \sec \phi$, $y = b \tan \phi$, $\phi$ excentric angle:

$\frac{x^2}{a^2} = \frac{y^2}{b^2}$, equation to asymptotes:

$xy = \frac{a^2 + b^2}{4}$, referred to asymptotes:

$\frac{x}{x_1} + \frac{y}{y_1} = 2$, tangent at $x_1$, $y_1$;

$x^2 - y^2 = a^2$, rectangular hyperbola.

## ANY CONIC SECTIONS.

Polar equation,

$$\frac{l}{r} = 1 - e \cos \theta :$$

to tangent,

$$\frac{l}{r} = \cos (\theta - \theta_1) - e \cos \theta,$$

general equation,

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where $B^2 - AC > = < 0$, as the curve is a hyperbola, parabola, or ellipse:

$$\left(\frac{x-a}{a}\right)^2 + \left(\frac{y-b}{b}\right)^2 = 1 + 2\lambda \frac{xy}{ab},$$

or, $\frac{x}{a} + \frac{y}{b} - 1 = \mu \left(\frac{xy}{ab}\right)^{\frac{1}{2}}$, referred to two tangents as axes.

$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$, parabola referred to two tangents.

# ANSWERS TO THE EXAMPLES.

---

I. 4. (i) 2, 1; (ii) $1\frac{4}{25}, \frac{22}{25}$; (iii) $-\frac{11}{7}, -\frac{5}{7}$; (iv) $x=y=2a$; (v) $c, 0$ or $0, c$; (vi) $\pm 2a, a$.

5. (i) $5a; \sqrt{5}b; a; \sqrt{10}a$. (ii) $\sqrt{37}a; \sqrt{3}b; \sqrt{}\left(1+\frac{\sin 2\alpha}{2}\right)^{\frac{1}{2}}a; \sqrt{7}a$.

6. $\sqrt{2}a; (h^2+4k^2)^{\frac{1}{2}}; \sqrt{2}(a-b); 10a; (\sec\alpha-\operatorname{cosec}\alpha)\{a^2(1+\sin 2\alpha)+b^2\}^{\frac{1}{2}}$.

7. $2a\cos\frac{\omega}{2}$; $a(1+\sin\omega)^{\frac{1}{2}}$; $\sec\frac{\omega}{2}\left\{a^2\left(\sin\frac{\omega}{2}-\cos\omega\cos\frac{\omega}{2}\right)^2 + b^2\left(1+\sin\omega\cos\frac{\omega}{2}\right)^2\right\}^{\frac{1}{2}}$.

8. $\{a^2+b^2-2ab\cos(\theta-\phi)\}^{\frac{1}{2}}$; $2a\sin\theta$; $2a\cos\theta$; $\sqrt{(5-2\sqrt{3})}\,a$.

9. $\frac{a}{2}, -\frac{a}{2}$; and $\frac{a}{3}, -\frac{2a}{3}$; $\frac{3h}{2}, -k$, and $\frac{4h}{3}, -\frac{k}{3}$; $\frac{a+b}{2}, \frac{a+b}{2}$ and $\frac{2a+b}{3}, \frac{2b+a}{3}$; $-6a, -2a$, and $-5a, -\frac{2a}{3}$; $a\operatorname{cosec}2\alpha$, $b(\sin\alpha+\cos\alpha)\operatorname{cosec}2\alpha$ and $\frac{2}{3}a\operatorname{cosec}2\alpha(1+\cos^2\alpha)$, $\frac{2}{3}b\operatorname{cosec}2\alpha(2\cos\alpha+\sin\alpha)$. 10. $\frac{ab}{2}$; $4a^2$; $\pm\frac{1}{2}(x_2y_1-x_1y_2)$; $\frac{1}{2}rr'\sin\theta$. 11. (1) $\frac{a}{2}, \frac{a}{2}\cot A$; $r, r\cot\frac{B}{2}$;

(2) $R\dfrac{\sin\frac{A+C}{2}}{\sin B}$, $R\dfrac{\sin\frac{A-B}{2}}{\sin A}$; $r\operatorname{cosec}A$, $r\operatorname{cosec}A$.

(3) $R, \frac{\pi}{2}-C$; $r, \frac{A}{2}$, where $R, r$ are the radii. 12. $\frac{2}{3}a, \frac{2}{3}b$.

II. 2. $\sqrt{2}x=c$. 3. $2x(x\cos 2\alpha - y\sin 2\alpha)=a^2$. 4. $y=0$. 5. $r\cos(\theta-\alpha)=a$; $r^2(2+\sin 2\theta)=2b^2$; $r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1$. 6. $r^2=(r\cos\theta+2a)^2$. 7. $(x^2+y^2)^2=a^2(x^2-y^2)$.

III. 19. $x + y\cos\theta = a$, $x + y\sec\theta = a$, where $(a, 0)$ is the given point. 20. $(x-a)\tan 30^\circ = y - b$.

21. $x + y = a + b$; $x = y$; $y = -k$; $(b\cos\phi - a\cos\theta)y - (b\sin\phi - a\sin\theta)x = ab\sin(\theta - \phi)$; $y = x + 1$; $y = 2x$.

22. $\frac{x}{a} + \frac{y}{b} = \lambda$, where $\lambda = \frac{10}{a}$, 2, 3, $\frac{h}{a} + \frac{k}{b}$, 2, 0, successively.

23. $Ax + By = Ax$. 24. If $\frac{c^2}{2}$ be the area of the given triangle, the equation is $\frac{x}{a} + \frac{y}{b} = \frac{c}{(ab)^{\frac{1}{2}}}$. 25. $y = mx + a(1 + m^2)^{\frac{1}{2}}$.

26. $x\sin\alpha + (y - a)\cos\alpha = 0$. 27. $x\cos\alpha - y\sin\alpha = b$.

28. $\frac{2ab}{(a^2 + b^2)^{\frac{1}{2}}}$; $\frac{b - ma}{(1 + m^2)^{\frac{1}{2}}}$; $\frac{7}{5}$; $\frac{-1}{\sqrt{2}}$; $\frac{-4}{\sqrt{10}}$. 29. $Bx = A(y - b)$.

31. $m^{\frac{1}{2}}y + \frac{x}{m^{\frac{1}{2}}} = a$, where $2a^2$ is the given area.

32. $(x - h)\cos(\alpha \pm \beta) + (y - k)\sin(\alpha \pm \beta) = 0$, where $\tan\beta = \frac{b}{a}$.

33. $x\cos(\alpha \pm \beta) + y\sin(\alpha \pm \beta) = 0$. 34. $b\sin(\alpha \pm \beta)$.

37. $y \pm \frac{x\sin(\omega \pm \theta)}{\sin\theta} = b$. 38. $y \pm \frac{x\sin\theta}{\sin(\omega \pm \theta)} = b$.

39. $r\{r_1\sin(\theta - \theta_1) - r_2\sin(\theta - \theta_2)\} = r_1 r_2\sin(\theta_2 - \theta_1)$;
$\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{r}$; $r(\sin\theta + \cos\theta) = a(2 - \sqrt{3})a$;
$\frac{\cos\theta}{b} + \frac{\sin\theta}{a} = \frac{1}{r}$. 40. $m = \frac{\sin(\omega - \theta)}{\sin\omega}$, $n = \frac{\sin\theta}{\sin\omega}$.

IV. 1. $2\sqrt{2}$. 2. $\frac{3}{\sqrt{13}}$. 3. $\frac{2\sqrt{10}}{5}$. 4. $\frac{11}{5}$.

5. $\frac{3ab}{(a^2 + b^2)^{\frac{1}{2}}}$. 6. $\frac{a^2 - b^2}{(a^2 + b^2)^{\frac{1}{2}}}$. 7. $\frac{Ah + Bk + C - D}{(A^2 + B^2)^{\frac{1}{2}}}$.

8. $\frac{c^2}{(h^2 + k^2)^{\frac{1}{2}}}$. 9. $\frac{h^2 + k^2 - c^2}{(h^2 + k^2)^{\frac{1}{2}}}$. 10. $(1 + m^2)^{\frac{1}{2}}a$.

11. $\dfrac{a^2b^2}{(a^4k^2+b^4h^2)^{\frac{1}{2}}}\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}-1\right)$. 12. $0, 1$; $\dfrac{30}{13}, -\dfrac{6}{13}$; $\dfrac{1}{5}, -\dfrac{4}{45}$; $-\dfrac{58}{25}, \dfrac{31}{25}$; $\dfrac{2b^2-a^2}{a^2+b^2}a, \dfrac{2a^2-b^2}{a^2+b^2}b$; $\dfrac{(Bh-Ak)B-A(C-D)}{(A^2+B^2)}, \dfrac{(Bh-Ak)A-B(C-D)}{A^2+B^2}$; $\dfrac{h}{(h^2+k)^{\frac{1}{2}}}, \dfrac{k}{(h^2+k^2)^{\frac{1}{2}}}$; $0, ma$; $\dfrac{(a^2-b^2)a^2k^2+a^4b^2}{a^4k^2+b^4h^2}h, \dfrac{(a^2-b^2)b^2h^2+a^2b^4}{a^4k^2+b^4h^2}k$.

13. $(a-b)\sin\alpha$; $\dfrac{5ab}{2(a^2+b^2)^{\frac{1}{2}}}$. 14. (1) and (2) intersect in $(4, -3)$; (1) and (3) in $(1, 0)$; (1) and (4) in $(6, -5)$; (2) and (3) in $\left(\dfrac{20}{17}, \dfrac{3}{11}\right)$; (2) and (4) in $\left(\dfrac{14}{3}, -4\right)$, and (3) and (4) in $\left(-\dfrac{2}{13}, -\dfrac{5}{13}\right)$. 15. $\dfrac{p'\sin\alpha-p\sin\beta}{\sin(\alpha-\beta)}, \dfrac{p\cos\beta-p'\cos\alpha}{\cos(\alpha-\beta)}$.

16. $a, p_1\operatorname{cosec}\alpha-a\cot\alpha$; $a, p_2\operatorname{cosec}\alpha-a\cot\alpha$; $b, p_2\operatorname{cosec}\alpha-b\cot\alpha$; $b, p_1\operatorname{cosec}\alpha-b\cot\alpha$. 17. $=0$; $\dfrac{a-b}{2}$, $x\cos\alpha+y\sin\alpha\pm\dfrac{(p_1-p_2)\{2x-(a-b)\}}{a-b}=\dfrac{p_1+p_2}{2}$; at their intersection $x\cos\alpha+y\sin\alpha=\dfrac{p_1+p_2}{2}$, and $x=\dfrac{a-b}{2}$, and the area is $(a\sim b)(p_1\sim p_2)\operatorname{cosec}\alpha$. 18. The three points being taken in the given order, the other point is $\dfrac{4h-a}{2}, \dfrac{4k-b}{2}$, and the equations to the diagonals are $\dfrac{x}{a}+\dfrac{y}{b}=1$, and $a(y-h)-2hy=b(x-h)-2kx$.

19. $\alpha$, and $\tan^{-1}\dfrac{ab-(ak+bh)}{(ah+bk)-(h^2+k^2)}$.

20. $\tan^{-1}\dfrac{a^2-b^2}{2ab}$; $\sqrt{3}(a^2-b^2)=2ab$.

21. $(\alpha-\beta)$, where $\tan\beta=\dfrac{a}{b}$. 22. $\tan^{-1}\dfrac{3}{4}$.

24. $x \sin \frac{A-B}{2} = (y-b) \sin \frac{A+B}{2}$; $x + y = 0$;

$$(x-c) \sin \frac{A+C}{2} = y \sin \frac{A-C}{2}; \quad x = y.$$

25. $\dfrac{\frac{x}{a_1} + \frac{y}{b_1} - 1}{\frac{h}{a_1} + \frac{k}{b_1} - 1} = \dfrac{\frac{x}{a_2} + \frac{y}{b_2} - 1}{\frac{h}{a_2} + \frac{k}{b_2} - 1}$.

26. $(A_1C_2 - A_2C_1)x + (B_1C_2 - B_2C_1)y = 0$.

27. $\frac{x}{b} - \frac{y}{a} = \frac{ma+b}{a-mb}$.

28. $(A_1B_2 - A_2B_1)(A_3x + B_3y) = C_1(A_2B_3 - A_3B_2) + C_2(A_3B_1 - A_1B_3)$.

29. $(1-m)p\left\{x \cos \frac{\alpha+\beta}{2} - y \cos \frac{\alpha+\beta}{2}\right\} - c \cos \frac{\alpha-\beta}{2}(x-y)$

$$= pc\left(\cos \frac{\alpha+\beta}{2} - \cos \frac{\alpha+\beta}{2}\right).$$

V. 1. $3a$, $2a$, $(a^2+b^2)^{\frac{1}{2}}$. 2. $a$, $-a$, $(a^2+b^2+c^2)^{\frac{1}{2}}$; $-\frac{a}{2}$, $-\frac{b}{2}$, $\frac{\{5(a^2+b^2)\}^{\frac{1}{2}}}{2}$; $\frac{3}{2}$, $2$, $\frac{3}{2}$; $\frac{a}{2}$, $0$, $\frac{a}{2}$; $0$, $\frac{b}{2}$, $\frac{b}{2}$; $\frac{a}{2}$, $0$, $\frac{\sqrt{5}}{2}a$; $\frac{a}{2}$, $\frac{b}{2}$, $\frac{(a^2+b^2)^{\frac{1}{2}}}{2}$. 3. $x^2 + y^2 - 2a(x-y) + a^2 = 0$.

4. $x^2 + y^2 = 2a(3x + 4y)$.

5. $x^2 + y^2 - 2(b+c)x - 2(b-c)y + 2b^2 + c^2 = 0$.

6. $x^2 + y^2 - ax - by = 0$.

7. $(x_2y_1 - x_1y_2)(x^2 + y^2) = \{(x_2^2y_1 - x_1^2y_2) - y_1y_2(y_1 - y_2)\}x$

$$+ \{x_2y_1^2 - x_1y_2^2 - x_1x_2(x_1 - x_2)\}y.$$

8. In equation (7) transpose the origin to the point $x_3$, $y_3$, and write $x - x_3$ for $x$, $y - y_3$ for $y$, &c. 9. $x^2 + y^2 = ax + by$.

10. See Art. 44.

11. $x^2 + y^2 + 4\frac{l^2a^2 - m^2b^2}{m^2 - l^2} = 4(mx + ly)\left(\frac{a^2 - b^2}{m^2 - l^2}\right)^{\frac{1}{2}}$.

12. $x^2 + y^2 \pm 2a(x \pm y) = 0$. 13. $x^2 + y^2 = \sqrt{2}ay$.

15. $\frac{\pi}{3}$; $\frac{a}{3}$, $\frac{a}{3}$, $\frac{2}{\sqrt{3}}a$. 16. $x^2 + 2xy \cos \omega + y^2 = hx + ky$.

17. $\frac{5\pi}{6}$, $4a$, $-2\sqrt{3}a$; $\sqrt{27+16\sqrt{3}}\,.\,a$.

18. $\cos\omega = \frac{a^2-b^2}{a^2}$; the coordinates of centre are

$$x = y = \frac{a^3}{2a^2-b^2}, \text{ and } r^2 = \frac{4a^2-b^2}{2a^2-b^2}a^2.$$

20. If $2l$ be the given length the equation is $r = (p^2+l^2)^{\frac{1}{2}}$.

25. $2\left(c^2 - \frac{a^2b^2}{a^2+b^2}\right)^{\frac{1}{2}}$.

VI. 2. $x = c$; $3x - 4y = 5c$; $ky - hx = c^2$.

3. $y\cos\alpha - x\sin\alpha = \pm c$, where $\alpha$ is the given angle.

4. $\frac{x}{a} + \frac{y}{b} = \pm\frac{c}{ab}(a^2+b^2)^{\frac{1}{2}}$. 5. $Bx - Ay = \pm(A^2+B^2)^{\frac{1}{2}}c$.

6. $c(y-b) = \pm(b^2-c^2)^{\frac{1}{2}}x$.

7. In the equation $x\cos\theta + y\sin\theta = c$, $\cos(\theta-\alpha) = \frac{\delta+c}{a}$.

8. In the same equation, if $a^2$ be the area of the triangle, $\sin 2\theta = \frac{c^2}{a^2}$.

9. $\frac{C^2}{A^2+B^2} = c^2$, $x\sin\alpha = \pm c$, $\frac{d^2}{l^2+m^2} = c^2$.

12. $a, b$; $2a, b$; $c, 0$; $0, c$, and $c(1+\cos\theta)$, $c(1+\sin\theta)$; $0, b$; 14. $a\cos\theta + b\sin\theta - p = (a^2+b^2)^{\frac{1}{2}}$.

19. $\theta = \frac{\pi}{2} + \alpha$, and $r\cos(\theta-\alpha) = l$.

20. $\theta = \frac{\pi}{4} + \frac{\alpha-\gamma}{2}$, $r = 2\cos\left(\frac{\pi}{4} + \frac{\alpha+\gamma}{2}\right)$.

VII. 1. $x + y = c$; $2x + 3y = c$; $(a+b)x + (a-b)y = c^2$.

2. $-\frac{Ac^2}{C}, -\frac{Bc^2}{C}$; $\frac{c^2}{2a}, \frac{c^2}{2b}$; $\frac{c^2\sin\theta}{x_1\sin\theta - y_1\cos\theta}, \frac{-c^2\cos\theta}{x_1\sin\theta - y_1\cos\theta}$; $0, \frac{c^2}{b}$; $\frac{c^2}{a}, 0$.

5. $x^2 + y^2 = c^2 - \dfrac{a^2}{2} + ax.$ 6. $x + y + 1 = 0.$ 7. $x = 0,\ y = 0.$

8. $\{4c^2 - 2(a-b)^2\}^{\frac{1}{2}}.$

10. $x + y = 3$; $2x - 4y + 5 = 0$; $4x - 2y = 1$; $\dfrac{7}{6}$, $\dfrac{11}{6}$.

14. $x \pm \sqrt{15}y = 8c$, and $3x \pm \sqrt{7}y = 8c$.

15. $x = \pm c$; $2cy - (a+c)c = \pm(a^2 + ac - 3c^2)^{\frac{1}{2}} x.$

16. $x = a + c.$ 17. A circle.

19. $\{(x-h)^2 + (y-k)^2\}(x\cos\alpha + y\sin\alpha - p)^2 = b^4$, where $(h, k)$ is the given point. 20. 22. 24. 25. 26. Circles.

VIII. 4. $4(2 \pm \sqrt{3})a.$

9. $\dfrac{\sqrt{5}-2}{4}a$, $\sqrt{\sqrt{5}-2}\,a$, $\dfrac{\sqrt{5}+2}{4}a$, $\sqrt{\sqrt{5}+2}\,a.$

14. $(x_1 x_2)^{\frac{1}{2}}$, $\dfrac{y_1 + y_2}{2}$. 15. Either extremity of the latus rectum. 21. $2hy - kx = hk \pm (k^2 - 4ah)^{\frac{1}{2}}.(x-h)$;

$(k^2 - 2ah - 2a^2)y - 2ak(x-a) = \pm(k^2 - 4ah)^{\frac{1}{2}}\{ky - 2a(x-a)\}$;

$\dfrac{y}{k} = \dfrac{x-a}{h}$. 24. $\{(c^2 + 4a^2)^{\frac{1}{2}} - c\}x - 2ac = \pm[2c\{(c^2 + 4a^2)^{\frac{1}{2}} - c\}]^{\frac{1}{2}}y.$

25. $(k^2a - 4x)(y - ka)^2 = 4ax^2.$

26. The parabola, $y^2 = 4ax + k^2a^2.$

27. $kx = a.$ 28. $(x-a)^2 + y^2 = \dfrac{a^2}{k^2}.$

29. $a\cot\alpha\cos\alpha$, $a\,\mathrm{cosec}\,\alpha.$ 31. The parabola $y^2 = a(x - 3a).$

35. $2ax^2 = (c - 2a)y^2.$ 40. A circle.

41. $r = l\cot\dfrac{\theta}{2}.$ 47. $r = 2l\,\mathrm{cosec}^2\,\theta.$ 49. $x = a.$

50, 51. Straight lines through the intersection of the diameter and tangent. 57—64. These loci are parabolas.

IX. 7. (i) $\dfrac{1}{\sqrt{2}}, \sqrt{2}$; (ii) $\dfrac{1}{2}, 3$; (iii) $\dfrac{1}{\sqrt{2}}, 1$;

(iv) $\left(\dfrac{b-a}{a}\right)^{\frac{1}{2}}, \dfrac{2a}{b^{\frac{1}{2}}}$; (v) $\left(\dfrac{2b}{a+b}\right)^{\frac{1}{2}}, \dfrac{2(a-b)}{(a+b)^{\frac{1}{2}}}$; (vi) $\dfrac{a-b}{(a^2+b^2)^{\frac{1}{2}}}$, $2\left(\dfrac{2ab}{a^2+b^2}\right)^{\frac{1}{2}}(a+b).$

8. $2,\ 2,\ \sqrt{2},\ 2\left\{\frac{b}{a}(a-b)\right\}^{\frac{1}{2}},\ 2(2b)^{\frac{1}{2}},\ \frac{a^2-b^2}{\{2ab(a^2+b^2)\}^{\frac{1}{2}}}$.

9. $\frac{1}{\sqrt{2}}$. 10. $\frac{\sqrt{13}-1}{2\sqrt{3}}$. 11. $\tan^{-1}\frac{ak}{bh}$.

12. $\tan^{-1}\frac{a}{b}$. 13. $\tan^{-1}\frac{b}{ae}$. 16. $\tan\theta=\frac{b}{a}$.

17. $\tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{2}}$. 18. $\frac{\pi}{4}$.

20. If $2l$ be the length, $\alpha$ the inclination, $a^2=l^2\,\frac{1-e^2\cos\alpha}{1-e^2}$;

$$b^2=l^2(1-e^2\cos^2\alpha).$$

21. $\beta$ being the inclination, $a^2=\frac{l^2b^2\sin^2\beta}{b^2-l^2\cos^2\beta}$.

22. $\tan^{-1}\frac{b}{a},\ \frac{\pi}{4}$. 25. $\pm\frac{ab}{(a^2+b^2)^{\frac{1}{2}}}$. 26. $\frac{1}{z^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}$.

27. $\frac{ab}{a^2+b^2}\{a\pm(2ab)^{\frac{1}{2}}+b\}$.

X. 1. $y\cos\theta-x\sin\theta=\pm(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}$.

2. If $c^2$ be the area of the given triangle, the equations to the tangents are $\left(\frac{x}{a}\pm\frac{y}{b}\right)\{(c^2+ab)^{\frac{1}{2}}\pm(c^2-ab)^{\frac{1}{2}}\}=2c$.

3. $\frac{x}{a}+\frac{y}{b}=\pm\sqrt{2}$. 4. $x\pm y=\pm\frac{(a^2+b^2)^{\frac{1}{2}}}{\sqrt{2}}$.

5. $x\cos\theta+y\sin\theta=(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}$

where $h\cos\theta+y\cos\theta-(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}=0$.

6. $\frac{a}{\cos\phi}+\frac{b}{\sin\phi}=l$, where $\phi$ is the excentric angle of the point of contact.

7. $a\sec\phi+b\,\mathrm{cosec}\,\phi+(a^2\sec^2\phi+b^2\,\mathrm{cosec}^2\,\phi)^{\frac{1}{2}}=2s$.

8. If $m$ be the ratio $\cos\phi=\frac{m-1}{m+1}\cdot\frac{1}{e}$.

9. $A^2a^2 + B^2b^2 = C^2, \ a^2m^2 + b^2 = c^2.$

10. $\dfrac{a^2 \cos \omega}{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)^{\frac{1}{2}}}, \ \dfrac{b^2 \sin \omega}{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)^{\frac{1}{2}}}.$

11. $x + \sqrt{2}y = \sqrt{6}.$ 12. $\dfrac{x}{a} + \dfrac{ey}{b} = \pm (1 + e^2)^{\frac{1}{2}}.$

18. $PG = b\,(1 - e^2 \cos^2 \phi)^{\frac{1}{2}}, \ CK = \dfrac{e^2 a \sin 2\phi}{2\,(1 - e^2 \cos^2 \phi)^{\frac{1}{2}}}.$

19. $\dfrac{2\,(a^4{y_1}^2 + b^4{x_1}^2)^{\frac{3}{2}}}{a^6{y_1}^2 + b^6{x_1}^2}$, or $\dfrac{2b\,(1 - e^2 \cos^2 \phi_1)^{\frac{3}{2}}}{1 - (2 - e^2)e^2 \cos^2 \phi_1}.$

20. $y = \dfrac{a}{bc}(x - c)\,(e^4a^2 - c^2)^{\frac{1}{2}}; \ eax = (y - c)\,(e^4a^4 - b^2c^2)^{\frac{1}{2}};$
in (111) substitute $(h, k)$ for $(x_1 y_1)$, this will give an equation of condition. 22. $e^2 \dfrac{a}{b}(a^2 - e^2{x_1}^2)^{\frac{1}{2}}.$

XI. 1. $b^2x + a^2y = 0$; $b^2x = a^2y$; $b^3y + a^3x = 0$; $by + ax = 0$; $a^2y \sin \theta + b^2x \cos \theta = 0$; $ay = bx.$

3. $\tan^2 \phi = \dfrac{a^2 - c^2}{c^2 - b^2}; \ \dfrac{y}{b}(a^2 - c^2)^{\frac{1}{2}} = \pm \dfrac{x}{a}(c^2 - b^2)^{\frac{1}{2}}.$

5. $\dfrac{y}{b}(3a + b)^{\frac{1}{2}} \pm \dfrac{x}{a}(3b + a)^{\frac{1}{2}} = 0.$

THE END.

---

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.

CAMBRIDGE,
*February*, 1867.

PUBLISHED BY

DEIGHTON, BELL, AND CO.
Agents to the University,
AND
BELL AND DALDY, LONDON.

NOW IN COURSE OF PUBLICATION.

Uniformly printed in Foolscap 8vo.

## Cambridge School and College Text Books,

*A Series of Elementary Treatises adapted for the Use of Students in the Universities, Schools, and Candidates for the Public Examinations.*

In order to secure a general harmony in the treatment, these works will be edited by Members of the University of Cambridge, and the methods and processes employed in University teaching will be followed.

Principles will be carefully explained, clearness and simplicity will be aimed at, and an endeavour will be made to avoid the extreme brevity which has so frequently made the Cambridge treatises too difficult to be used by those who have not the advantage of a private Tutor. Copious examples will be added.

*Now Ready.*

ARITHMETIC for the use of Schools and Colleges. By A. WRIGLEY, M.A. Professor of Mathematics in the late Royal Military College, Addiscombe. 3*s*. 6*d*. cl.

ELEMENTARY TRIGONOMETRY. By T. P. HUDSON, M.A. Fellow of Trinity College. 3*s*. 6*d*. cl.

ELEMENTARY STATICS. By the Very Rev. H. GOODWIN, D.D. DEAN OF ELY. 3*s*. cl.

ELEMENTARY DYNAMICS. By the Very Rev. H. GOODWIN, D.D. DEAN OF ELY. 3*s*. cl.

[*Continued*]

ELEMENTARY HYDROSTATICS. By W. H. BESANT, M.A., Late Fellow of St John's College. 4*s.* cl.

MENSURATION, An Elementary Treatise on. By B. T. MOORE, M.A., Fellow of Pembroke College, Professor of Mathematics, Royal Staff College, Sandhurst. With numerous Examples. 5*s.*

ELEMENTARY GEOMETRICAL CONIC SECTIONS. By W. H. BESANT, M.A., Late Fellow of St John's College. [*In the Press.*

ELEMENTARY CHEMISTRY. By G. D. LIVEING, M.A., Late Fellow of St John's College. *Preparing.*

*Now in course of Publication,*

## Cambridge Greek and Latin Texts,

### CAREFULLY REPRINTED FROM THE BEST EDITIONS.

THIS series is intended to supply for the use of Schools and Students cheap and accurate editions of the Classics, which shall be superior in mechanical execution to the small German editions now current in this country, and more convenient in form.

The texts of the "*Bibliotheca Classica*" and "*Grammar-School Classics,*" so far as they have been published, will be adopted. These editions have taken their place amongst scholars as valuable contributions to the Classical Literature of this country, and are admitted to be good examples of the judicious and practical nature of English scholarship; and as the editors have formed their texts from a careful examination of the best editions extant, it is believed that no texts better for general use can be found.

The volumes are well printed at the Cambridge University Press, in a 16mo. size, and are issued at short intervals.

*The following are now ready.*

NOVUM TESTAMENTUM Graecum, Textus Stephanici, 1550. Accedunt variae lectiones editionum Bezae, Elzeviri, Lachmanni, Tischendorfii, Tregellesii. Curante F. H. Scrivener, A.M. *4s. 6d.*

*An Edition on writing paper, for Notes.* 4to. *half-bound.* 12*s.*

AESCHYLUS, ex novissima recensione F. A. Paley, A.M. Price 3*s.*

CAESAR DE BELLO GALLICO, recensuit G. Long, A.M. 2*s.*

CICERO DE SENECTUTE ET DE AMICITIA ET EPISTOLAE SELECTAE, recensuit G. Long, A.M. 1*s.* 6*d.*

CICERONIS ORATIONES. Vol. I. Recensuit G. Long, A.M. 3*s.* 6*d.*

EURIPIDES, ex recensione F. A. Paley, A.M. Vol. i. 3*s.* 6*d.* Vol. ii. 3*s.* 6*d.* Vol. iii. 3*s.* 6*d.*

HERODOTUS, recensuit J. W. Blakesley, S.T.B. 2 Vols. 3*s.* 6*d.* each Vol.

HORATIUS, ex recensione A. J. Macleane, A.M. Price 2*s.* 6*d.*

JUVENAL ET PERSIUS, ex recensione A. J. Macleane, A.M. 1*s.* 6*d.*

LUCRETIUS, recognovit H. A. J. Munro, A.M. 2*s.* 6*d.*

SALLUSTI CATILINA ET JUGURTHA, ex recensione G. Long, A.M. 1*s.* 6*d.*

THUCYDIDES, recensuit J. G. Donaldson, S.T.P 2 Vols. 3*s.* 6*d.* each Vol.

VERGILIUS, ex recensione J. Conington, A.M. 3*s.* 6*d.*

XENOPHONTIS EXPEDITIO CYRI, recensuit J. F. Macmichael, A.B. 2*s.* 6*d.*

*OTHERS IN PREPARATION.*

## ARITHMETIC AND ALGEBRA.

Arithmetic for the use of Schools and Colleges. By A. WRIGLEY, M.A., Professor of Mathematics in the late Royal Military College, Addiscombe. 8*s*. 6*d*.

Principles and Practice of Arithmetic. By the Rev. J. HIND. Ninth Edition, with Questions, 4*s*. 6*d*.

⁂ KEY, with Questions for Examination. Second Edition. 5*s*.

A Progressive Course of Examples in Arithmetic. With Answers. By the Rev. JAMES WATSON, M.A., of Corpus Christi College, Cambridge, and formerly Senior Mathematical Master of the Ordnance School, Carshalton. *Second Edition, revised and corrected.* Fcp. 8vo. 2*s*. 6*d*.

Principles and Practice of Arithmetical Algebra, with Examples. By the Rev. J. HIND. Third Edition. 12mo. 5*s*.

Elements of Algebra. By the Rev. J. HIND. Sixth Edition, revised. 540 pp. 8vo. 10*s*. 6*d*.

Treatise on the Theory of Algebraical Equations. By the Rev. J. HYMERS, D.D. Third Edition. 8vo. 10*s*. 6*d*.

## TRIGONOMETRY.

Trigonometry required for the Additional Subject for Honours at the Previous Examination, according to the new scheme sanctioned by the Senate June 1865. By J. MᶜDOWELL, M.A., Pembroke College. Crown 8vo. 3*s*. 6*d*.

Elementary Trigonometry. By T. P. HUDSON, M.A., Fellow of Trinity College. 3*s*. 6*d*.

Elements of Plane and Spherical Trigonometry. By the Rev. J. HIND. Fifth Edition. 12mo. 6*s*.

Syllabus of a Course of Lectures upon Trigonometry and the Application of Algebra to Geometry. 8vo. 7*s*. 6*d*.

## MECHANICS AND HYDROSTATICS.

Elementary Hydrostatics. By W. H. Besant, M.A., Late Fellow of St John's College. Fcp. 8vo. 4*s.*

Elementary Hydrostatics for Junior Students. By R. POTTER, M.A. late Fellow of Queens' College, Cambridge, Professor of Natural Philosophy and Astronomy in University College, London. 7*s.* 6*d.*

The Propositions in Mechanics and Hydrostatics which are required for those who are not Candidates for Honours. By A. C. BARRETT, M.A. Third Edition. Crown 8vo. 6*s.*

Mechanical Euclid. Containing the Elements of Mechanics and Hydrostatics. By the late W. WHEWELL, D.D. Fifth Edition. 5*s.*

Elementary Statics. By the Very Rev. H. Goodwin, D.D. Dean of Ely. Fcp. 8vo, cloth, 8*s.*

Elementary Dynamics. By the Very Rev. H. GOODWIN, D.D. Dean of Ely. Fcp. 8vo, cloth, 8*s.*

A Treatise on Statics. By the Rev. S. Earnshaw, M.A. Fourth Edition. 8vo. 10*s.*

Dynamics, or, a Treatise on Motion. By the Rev. S. EARNSHAW. Third Edition. 8vo. 14*s.*

A Treatise on the Dynamics of a Rigid Body. By the Rev. W. N. GRIFFIN. 8vo. 6*s.* 6*d.*

⁂ SOLUTIONS OF THE EXAMPLES. 8vo.

Problems in illustration of the Principles of Theoretical Mechanics. By W. WALTON, M.A. Second Edition. 8vo. 18s.

Treatise on the Motion of a Single Particle and of two Particles acting on one another. By A. SANDEMAN. 8vo. 8s. 6d.

Of Motion. An Elementary Treatise. By the Rev. J. R. LUNN, M.A. Fellow and Lady Sadleir's Lecturer of St John's College. 8vo. 7s. 6d.

> Chapter I. General principles of velocity and acceleration. Chapter II. Of the motion of a point in general. Analytical expressions for velocities and accelerations in certain directions. Chapter III. Of the motion of a point affected by a constant acceleration, the direction of which is always the same. Chapter IV. Of the motion of a point affected by an acceleration, the direction of which always passes through a fixed point. Chapter V. Of matter and force. Chapter VI. Of the dynamical laws of force, commonly called the laws of motion. Chapter VII. Of certain cases of free motion in nature. Chapter VIII. Of constrained motion of particles. Chapter IX. Of impulses and collision of particles. *Appendix*. Of the Cycloid.

Treatise on Hydrostatics and Hydrodynamics. By W. H. BESANT, M.A. 8vo. 9s.

The Principles of Hydrostatics. By T. WEBSTER, M.A. 8vo. 7s. 6d.

Problems in illustration of the Principles of Theoretical Hydrostatics and Hydrodynamics. By W. WALTON, M.A. 8vo. 10s. 6d.

Collection of Elementary Problems in Statics and Dynamics. Designed for Candidates for Honours, first three days. By W. WALTON, M.A. 8vo. 10s. 6d.

## CONIC SECTIONS AND ANALYTICAL GEOMETRY.

Trilinear Co-ordinates, and other methods of *Modern Analytical Geometry* of Two Dimensions. By the Rev. W. ALLEN WHITWORTH, M.A., Professor of Mathematics in Queen's College, Liverpool, and late Scholar of St John's College, Cambridge. 8vo. 16s.

An Introduction to Plane Co-ordinate Geometry. By W. P. TURNBULL, B.A. Fellow of Trinity College. [*In the Press.*

Elementary Geometrical Conic Sections. By W. H. BESANT, M.A., Late Fellow of St John's College. [*In the Press.*

Conic Sections. Their principal Properties proved Geometrically. By the late W. WHEWELL, D.D. Master of Trinity. Third Edition. 8vo. 2*s*. 6*d*.

The Geometrical Construction of a Conic Section. By the Rev. T. GASKIN. 8vo. 3*s*.

Treatise on Conic Sections. By the Rev. J. HYMERS, D.D. Third Edition. 8vo. 9*s*.

A Treatise on the Application of Analysis to Solid Geometry. By D. F. GREGORY, M.A. and W. WALTON, M.A. Second Edition. 8vo. 12*s*.

The Elements of Conic Sections. By J. D. HUSTLER, B.D. Fourth Edition. 8vo. 4*s*. 6*d*.

Elementary Treatise on Solid Geometry. By W. S. ALDIS, M.A. 8vo. 8*s*.

A Treatise on Plane Co-ordinate Geometry. By the Rev. M. O'BRIEN. 8vo. 9*s*.

A Treatise on Analytical Geometry of Three Dimensions. By J. HYMERS, D.D. Third Edition. 8vo. 10*s*. 6*d*.

Problems in illustration of the Principles of Plane Co-ordinate Geometry. By W. WALTON, M.A. 8vo. 16*s*.

---

## DIFFERENTIAL AND INTEGRAL CALCULUS.

An Elementary Treatise on the Differential Calculus. By W. H. MILLER, M.A. Third Edition. 8vo. 6*s*.

Treatise on the Differential Calculus. By W. WALTON, M.A. 8vo. 10*s*. 6*d*.

A Treatise on the Integral Calculus. By the Rev. **J. HYMERS, D.D. 8vo. 10s. 6d.**

Geometrical Illustrations of the Differential Calculus. **By M. B. PULL. 8vo. 2s. 6d.**

Examples of the Principles of the Differential and **Integral Calculus. Collected by D. F. GREGORY. Second Edition. Edited by W. WALTON, M.A. 8vo. 18s.**

## ASTRONOMY.

Elementary Treatise on Astronomy for the use of **Colleges and Schools, and for Students preparing for the three days' Examination in the Senate-House. By P. T. MAIN, B.A. Fellow of St. John's College. 8vo. 7s. 6d.**

Practical and Spherical Astronomy for the use **chiefly of Students in the Universities. By the Rev. R. MAIN, M.A., Radcliffe Observer, Oxford. 8vo. 14s.**

Brünnow's Spherical Astronomy. Translated by **the Rev. R. MAIN, M.A. F.R.S. Radcliffe Observer. Part I. Including the Chapters on Parallax, Refraction, Aberration, Precession, and Nutation. 8vo. 8s. 6d.**

Elementary Chapters on Astronomy from the **"Astronomie Physique" of Biot. By the Very Rev. HARVEY GOODWIN, D.D. Dean of Ely. 8vo. 3s. 6d.**

> **"They were translated with a different intention, but the admirable precision and clearness of description which characterise them led me to think that the publication of them would make a useful addition to our present list of elementary books."—*Translator's Preface.***

Lectures on Practical Astronomy. By the Rev. J. **CHALLIS, M.A., F.R.S., F.R.A.S., Plumian Professor of the University.** *[Preparing.*

---

Exercises on Euclid and in Modern Geometry, **containing Applications of the Principles and Processes of Modern Pure Geometry. By J. McDOWELL, B.A., F.R.A.S., Pembroke College. pp. xxxi, 300. Crown 8vo. 8*s.* 6*d.***

Elementary Course of Mathematics. Designed **principally for Students of the University of Cambridge. By the Very Rev. HARVEY GOODWIN, D.D., Dean of Ely. Sixth Edition, revised and enlarged by P. T. MAIN, M.A., Fellow of St John's College, Cambridge. 8vo. 16*s.***

Problems and Examples, adapted to the "Elementary Course of Mathematics." **By HARVEY GOODWIN, D.D. Dean of Ely. Third Edition, revised, with Additional Examples in Conic Sections and Newton. By THOMAS G. VYVYAN, M.A. Fellow of Gonville and Caius College. 8vo. 5*s.***

Solutions of Goodwin's Collection of Problems and **Examples. By W. W. HUTT, M.A. late Fellow of Gonville and Caius College. Third Edition, revised and enlarged. By the Rev. T. G. VYVYAN, M.A. 8vo. 9*s.***

Collection of Examples and Problems in Arithmetic, **Algebra, Geometry, Logarithms, Trigonometry, Conic Sections, Mechanics, &c. with Answers and Occasional Hints. By the Rev. A. WRIGLEY. Sixth Edition. 8vo. 8*s.* 6*d.***

A Companion to Wrigley's Collection of Examples **and Problems, being Illustrations of Mathematical Processes and Methods of Solution. By J. PLATTS, Esq., and the Rev. A. WRIGLEY, M.A. 8vo. 15*s.***

Newton's Principia. First Three Sections, with **Appendix, and the Ninth and Eleventh Sections. By the Rev. J. H. EVANS, M.A. Fourth Edition. 8vo. 6*s.***

Series of Figures Illustrative of Geometrical Optics. **From SCHELLBACH. By the Rev. W. B. HOPKINS. Plates Folio. 10*s.* 6*d.***

A Treatise on Crystallography. By W. H. MILLER, **M.A. 8vo. 7*s.* 6*d.***

**A Tract on Crystallography, designed for Students in the University.** By W. H. Miller, M.A. Professor of Mineralogy in the University of Cambridge. 8vo. 5*s.*

**Physical Optics, Part II. The Corpuscular Theory** of Light discussed Mathematically. By RICHARD POTTER, M.A. Late Fellow of Queens' College, Cambridge, Professor of Natural Philosophy and Astronomy in University College, London. 7*s.* 6*d.*

---

**The Greek Testament: with a critically revised** Text; a Digest of Various Readings; Marginal References to Verbal and Idiomatic Usage; Prolegomena; and a Critical and Exegetical Commentary. For the use of Theological Students and Ministers. By HENRY ALFORD, D.D. Dean of Canterbury.

Vol. I. Fifth Edition, containing the Four Gospels. 1*l.* 8*s.*

Vol. II. Fifth Edition, containing the Acts of the Apostles, the Epistles to the Romans and Corinthians. 1*l.* 4*s.*

Vol. III. Fourth Edition, containing the Epistles to the Galatians, Ephesians, Philippians, Colossians, Thessalonians,—to Timotheus, Titus and Philemon. 18*s.*

Vol. IV. Part I. Third Edition, containing the Epistle to the Hebrews, and the Catholic Epistle of St James and St Peter. 18*s.*

Vol. IV. Part II. Second Edition, containing the Epistles of St John and St Jude, and the Revelation. 14*s.*

**Codex Bezæ Cantabrigiensis. Edited with Prolegomena,** Notes, and Facsimiles. By F. H. SCRIVENER, M.A. Small 4to. 26*s.*

**Wieseler's Chronological Synopsis of the Four Gospels.** Translated by the Rev. E. VENABLES, M.A. 8vo. 13*s.*

> "This opportunity may properly be taken of especially recommending to every thoughtful student this able treatise on the succession of events in the Gospel history . . . . . . . A translation of it would be a very welcome aid to the general reader."—*Bp. Ellicott's Lectures on the Life of our Lord.*

**Bentleii Critica Sacra.**

Notes on the Greek and Latin Text of the New Testament, extracted from the Bentley MSS. in Trinity College Library. With the Abbé Rulotta's Collation of the Vatican MS., a specimen of Bentley's intended Edition, and an account of all his Collations. Edited, with the permission of the Master and Seniors, by the Rev. A. A. ELLIS M.A., late Fellow of Trinity College, Cambridge. 8vo. 8*s.* 6*d.*

**A Companion to the New Testament. Designed** for the use of Theological Students and the Upper Forms in Schools. By A. C. BARRETT, M.A. Caius College. Fcap. 8vo. *5s.*

**A general Introduction to the Apostolic Epistles,** With a Table of St Paul's Travels, and an Essay on the State after Death. *Second Edition, enlarged.* To which are added a Few Words on the Athanasian Creed, on Justification by Faith, and on the Ninth and Seventeenth Articles of the Church of England. By A BISHOP'S CHAPLAIN. 8vo. *8s. 6d.*

**Butler's Three Sermons on Human Nature, and** Dissertation on Virtue. Edited by the late W. WHEWELL, D.D. With a Preface and a Syllabus of the Work. Third Edition. Fcp. 8vo. *3s. 6d.*

**An Historical and Explanatory Treatise on the** Book of Common Prayer. By W. G. HUMPHRY, B.D. Third and Cheaper Edition, revised and enlarged. Fcap. 8vo. *4s. 6d.*

**Annotations on the Acts of the Apostles. Original** and selected. Designed principally for the use of Candidates for the Ordinary B.A. Degree, Students for Holy Orders, &c., with College and Senate-House Examination Papers. By the Rev. T. R. MASKEW, M.A. Second Edition, enlarged. 12mo. *5s.*

**An Analysis of the Exposition of the Creed, written** by the Right Reverend Father in God, J. PEARSON, D.D. late Lord Bishop of Chester. Compiled, with some additional matter occasionally interspersed, for the use of Students of Bishop's College, Calcutta. By W. H. MILL, D.D. Third Edition, revised and corrected. 8vo. *5s.*

**Hints for some Improvements in the Authorised** Version of the New Testament. By the late J. SCHOLEFIELD, M.A. Fourth Edition. Fcap. 8vo. *4s.*

**A Plain Introduction to the Criticism of the New** Testament. With 40 facsimiles from Ancient Manuscripts. For the use of Biblical Students. By F. H. SCRIVENER, M.A. Trinity College, Cambridge. 8vo. *15s.*

**The Apology of Tertullian. With English Notes** and a Preface, intended as an Introduction to the Study of Patristical and Ecclesiastical Latinity. By H. A. WOODHAM, LL.D. Second Edition. 8vo. *8s. 6d.*

Æschylus, Translated into English Prose, by F. A. PALEY, M.A. Editor of the Greek Text. 8vo. 7*s*. 6*d*.

Aristophanis Comœdiæ Undecim, cum Notis et Onomastico, by the Rev. H. A. HOLDEN, LL.D., Head-Master of Ipswich School, late Fellow and Assistant Tutor of Trinity College, Cambridge. Second Edition. 8vo. 15*s*.

The Plays separately, 1*s*., 1*s*. 6*d*. and 2*s*. each.

NOTES, 4*s*.

Demosthenes, the Oration against the Law of Leptines, with English Notes, and a Translation of Wolfe's Prolegomena. Edited by B. W. BEATSON, M.A. Fellow of Pembroke College, Cambridge. Second Edition. Small 8vo. 6*s*.

Demosthenes de Falsa Legatione. Third Edition, carefully revised. By R. SHILLETO, A.M. 8vo. 8*s*. 6*d*.

Demosthenes. Select Private Orations of. After the text of DINDORF, with the Various Readings of REISKE and BEKKER. With English Notes. For the use of Schools. By C. T. PENROSE, A.M. Second Edition. Revised and corrected. 12mo. 4*s*.

Euripides. Fabulæ Quatuor, scilicet, Hippolytus Coronifer, Alcestis, Iphigenia in Aulide, Iphigenia in Tauris. Ad fidem Manuscriptorum ac veterum Editionum emendavit et Annotationibus instruxit J. H. MONK, S.T.P. Editio nova. 8vo. 12*s*.

*Separately*—Hippolytus, 8vo, cloth, 5*s*.; Alcestis, 8vo, sewed, 4*s*. 6*d*.

Lucretius. With a literal Translation and Notes Critical and Explanatory, by the Rev. H. A. J. MUNRO, M.A. Fellow of Trinity College, Cambridge. Second Edition, revised throughout. 2 Vols. 8vo. Vol. I. Text, 16*s*. Vol. II. Translation, 6*s*. May be had separately.

Plato. The Gorgias, literally translated, with an Introductory Essay, containing a Summary of the Argument. By E. M. COPE, M.A. Fellow of Trinity College, Cambridge. 8vo. 7*s*.

Plato, The Protagoras. The Greek Text, with English Notes. By W. WAYTE, M.A. 8vo. 5*s*. 6*d*.

Plautus. Aulularia. With notes, Critical and Exegetical, and an Introduction on the Plautian Metres and Prosody. By Dr WM. WAGNER. 8vo. 9*s*.

Plautus. Aulularia. Ad fidem Codicum qui in Bibliotheca Musei Britannici exstant aliorumque nonnullorum recensuit, Notisque et Glossario locuplete instruxit J. HILDYARD, A.M. Editio altera. 8vo. 7*s*. 6*d*.

**Plautus. Menæchmei. Ad fidem Codicum qui** in Bibliotheca Musei Britannici exstant aliorumque nonnullorum recensuit, Notisque et Glossario locuplete instruxit J. HILDYARD, A.M. Editio altera. 7*s*. 6*d*.

**Propertius. The Elegies of. With English Notes,** and a Preface on the State of Latin Scholarship. By F. A. PALEY, M.A. With copious Indices. 10*s*. 6*d*.

**Verse-Translations from Propertius, Book V. With** a Revised Latin Text, and Brief English Notes. By F. A. PALEY, M.A. Editor of Propertius, Ovid's Fasti, &c. Fcp. 8vo. 3*s*.

**Theocritus, recensuit brevi commentario instruxit** F. A. PALEY, M.A. Crown 8vo. 4*s*. 6*d*.

**A Complete Greek Grammar. For the use of** Students. By the late J. W. DONALDSON, D.D. Third Edition, considerably enlarged. 8vo. 16*s*.

Without being formally based on any German Work, it has been written with constant reference to the latest and most esteemed of Greek Grammars used on the Continent.

**A Complete Latin Grammar. For the use of** Students. By the late J. W. DONALDSON, D.D. Second Edition, considerably enlarged. 8vo. 14*s*.

The enlarged Edition of the Latin Grammar has been prepared with the same object as the corresponding work on the Greek language. It is, however, especially designed to serve as a convenient handbook for those students who wish to acquire the habit of writing Latin; and with this view it is furnished with an Antibarbarus, with a full discussion of the most important synonyms, and with a variety of information not generally contained in works of this description.

**Varronianus. A Critical and Historical Introduction** to the Ethnography of Ancient Italy, and to the Philological Study of the Latin Language. By the late J. W. DONALDSON, D.D. Third Edition, revised and considerably enlarged. 8vo. 16*s*.

**The Theatre of the Greeks. A Treatise on the** History and Exhibition of the Greek Drama: with various Supplements. By the late J. W. DONALDSON, D.D. *Seventh Edition*, revised, enlarged, and in part remodelled, with numerous illustrations from the best ancient authorities. 8vo. 14*s*.

**Classical Scholarship and Classical Learning considered** with especial reference to Competitive Tests and University Teaching. A Practical Essay on Liberal Education. By the late J. W. DONALDSON, D.D. Crown 8vo. 5*s*.

**Sophocles. The Œdipus Coloneus of, with Notes,** intended principally to explain and defend the Text of the manuscripts as opposed to conjectural emendations. By the Rev. C. E. PALMER, M.A. 9*s.*

**Tacitus (C.). Opera, ad Codices antiquissimos** exacta et emendata, Commentario critico et exegetico illustrata. 4 vols. 8vo. Edidit F. RITTER, Prof. Bonnensis. Reduced to 14*s.*

**Translations into English and Latin. By C. S.** CALVERLEY, late Fellow of Christ's College, Cambridge. Small 8vo. 7*s.* 6*d.*

**P. Virgilii Maronis Opera edidit et syllabarum** quantitates novo eo que facili modo notavit Thomas Jarrett, M.A. Linguæ Hebrææ apud Cantabrigienses Professor Regius. 12*s.*

**Arundines Cami: sive Musarum Cantabrigiensium** Lusus Canori. Collegit atque ed. H. DRURY, A.M. Editio quinta. Cr. 8vo. 7*s.* 6*d.*

**Foliorum Silvula. Part the first. Being Passages** for Translation into Latin Elegiac and Heroic Verse. Edited with Notes by the Rev. HUBERT A. HOLDEN, LL.D., Head Master of Queen Elizabeth School, Ipswich. Late Fellow of Trinity College, Cambridge. Fourth Edition. Post 8vo. 7*s.* 6*d.*

**Foliorum Silvula. Part II. Being Select Passages** for Translation into Latin Lyric and Comic Iambic Verse. Arranged and edited by the Rev. HUBERT A. HOLDEN, LL.D. Third Edition. Post 8vo. 5*s.*

**Foliorum Silvula. Part III. Being Select Passages** for Translation into Greek Verse. Edited with Notes by the Rev. HUBERT A. HOLDEN, LL.D. Third Edition. Post 8vo. 8*s.*

**Folia Silvulæ, sive Eclogæ Poetarum Anglicorum** in Latinum et Græcum conversæ quas disposuit, HUBERTUS A. HOLDEN, LL.D. Volumen Prius. Continens Fasciculos I. II. 8vo. 10*s.* 6*d.*

**Foliorum Centuriæ. Being Select Passages for** Translation into Latin and Greek Prose. Arranged and edited by the Rev. HUBERT A. HOLDEN, LL.D. Third Edition. Post 8vo. 8*s.*

Kennedy (Rev. Dr). Progressive Exercises in Greek Tragic Senarii, followed by a Selection from the Greek Verses of Shrewsbury School, and prefaced by a short Account of the Iambic Metre and Style of Greek Tragedy. For the use of Schools and Private Students. Second Edition, altered and revised. 8vo. 8*s.*

---

Accidence Papers set in the Previous Examination, December, 1866. 12mo. 6*d.*

Cambridge Examination Papers. Being a Supplement to the Cambridge University Calendar, 1859. 12mo. 5*s.*

Containing those set for the Tyrwhitt's Hebrew Scholarships.—Theological Examinations.—Carus Prize.—Crosse Scholarships.—Mathematical Tripos.—The Ordinary B.A. Degree.—Smith's Prize.—University Scholarships.—Classical Tripos.—Moral Sciences Tripos.—Chancellor's Legal Medals.—Chancellor's Medals.—Bell's Scholarships.—Natural Sciences Tripos.—Previous Examination.—Theological Examination. With Lists of Ordinary Degrees, and of those who have passed the Previous and Theological Examinations.

*The Examination Papers of* 1856, *price* 2*s.* 6*d.*, 1857 *and* 8, 3*s.* 6*d. each, may still be had.*

A Manual of the Roman Civil Law, arranged according to the Syllabus of Dr HALLIFAX. By G. LEAPINGWELL, LL.D. Designed for the use of Students in the Universities and Inns of Court. 8vo. 12*s.*

The Mathematical and other Writings of ROBERT LESLIE ELLIS, M.A., late Fellow of Trinity College, Cambridge. Edited by WILLIAM WALTON, M.A. Trinity College, with a Biographical Memoir by the Very Reverend HARVEY GOODWIN, D.D. Dean of Ely. 8vo. 16*s.*

Lectures on the History of Moral Philosophy in England. By the late Rev. W. WHEWELL, D.D. Master of Trinity College, Cambridge. New and Improved Edition, with Additional Lectures. Crown 8vo. 8*s.*

*The Additional Lectures are printed separately in Octavo for the convenience of those who have purchased the former Edition. Price* 3*s.* 6*d.*

A Concise Grammar of the Arabic Language. Revised by SHEIKH ALI NADY EL BARRANY. By W. J. BEAMONT, M.A. Fellow of Trinity College, Cambridge, and Incumbent of St Michael's, Cambridge, sometime Principal of the English College, Jerusalem. Price 7*s.*

A Syriac Grammar. By G. PHILLIPS, D.D., President of Queens' College. Third Edition, revised and enlarged. 8vo. 7*s.* 6*d.*

*Now ready, fcap. 8vo. price 5s.*

## The Student's Guide to the University of Cambridge.

*SECOND EDITION, REVISED AND CORRECTED IN ACCORDANCE WITH THE RECENT REGULATIONS.*

### Contents.

INTRODUCTION, by J. R. SEELEY, M.A.
ON UNIVERSITY EXPENSES, by the Rev. H. LATHAM, M.A.
ON THE CHOICE OF A COLLEGE, by J. R. SEELEY, M.A.
ON THE COURSE OF READING FOR THE MATHEMATICAL TRIPOS, by the Rev. W. M. CAMPION, B.D.
ON THE COURSE OF READING FOR THE CLASSICAL TRIPOS, by the Rev. R. BURN, M.A.
ON THE COURSE OF READING FOR THE MORAL SCIENCES TRIPOS, by the Rev. J. B. MAYOR, M.A.
ON THE COURSE OF READING FOR THE NATURAL SCIENCES TRIPOS, by Professor LIVEING, M.A.
ON LAW STUDIES AND LAW DEGREES, by Professor J. T. ABDY, LL.D.
MEDICAL STUDY AND DEGREES, by G. M. HUMPHRY, M.D.
ON THEOLOGICAL EXAMINATIONS, by Professor E. HAROLD BROWNE, B.D.
THE ORDINARY (OR POLL) DEGREE, by the Rev. J. R. LUMBY, M.A.
EXAMINATIONS FOR THE CIVIL SERVICE OF INDIA, by the Rev. H. LATHAM, M.A.
LOCAL EXAMINATIONS OF THE UNIVERSITY, by H. J. ROBY, M.A.
DIPLOMATIC SERVICE.
DETAILED ACCOUNT OF THE SEVERAL COLLEGES.

"Partly with the view of assisting parents, guardians, schoolmasters, and students intending to enter their names at the University—partly also for the benefit of undergraduates themselves—a very complete, though concise, volume has just been issued, which leaves little or nothing to be desired. For lucid arrangement, and a rigid adherence to what is positively useful, we know of few manuals that could compete with this Student's Guide. It reflects no little credit on the University to which it supplies an unpretending, but complete, introduction."—SATURDAY REVIEW.

---

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.